ALGEBRA FOR SCHOOLS

PART I

WITH ANSWERS

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ALGEBRA FOR SCHOOLS

PART I

BY

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PREFACE.

THIS book is intended to meet the needs of those who require a concise Algebra, written on modern lines, but yet retaining the best features of the older text-books.

It covers the usual school syllabus, from the beginnings of the subject to the work required for the entrance examinations of the various Universities.

The present volume, which forms Part I. of the course, is up to the standard of the School Certificate in England and Wales, and contains the material required for the Intermediate Certificate Examination of the Scottish Education Department.

We have tried to develop the principles of Algebra in a direct and simple manner, making no attempt at rigour, but trusting to the pupils' knowledge of arithmetic, and to their intuitions and every-day experiences. The expansion of Arithmetic into Algebra has been effected in the first three chapters, the opening chapter dealing with the evaluation of simple formulæ, in order to familiarise the pupils with the use of letters; the second dealing with simple arithmetical operations; and the third introducing Directed Numbers.

Problems involving equations of the first degree have been treated after simple simultaneous equations, in order that the pupil may have the free choice of the use of one letter or of two in solving a problem.

The first chapter on Graphs, being of a purely arithmetical nature, may be taken before Chapter III., if so desired but our experience is that the average beginner

is so disheartened by the mechanical difficulties that he fails to grasp the fundamental idea of the inter-dependence of the two related quantities.

The rather unusual course has been taken of grouping in three successive chapters the operations of "Multiplication at Sight," "Factorisation of the Quadratic Expression," and "Solution of Quadratic Equations by means of Factors," on the idea that these form a natural sequence.

Bearing in mind the value and importance of example-working, we have devoted much care and attention to the collecting and the grading of a plentiful supply of exercises on the subject-matter of each chapter, and of a large number of Revision Papers at the end of the book. In the blocks of examples in each chapter, whenever possible, applications of the process in question have been set as a corrective to mere mechanical dexterity.

Our best thanks are due to the General Editor, Professor W. P. Milne, whose criticisms and advice throughout the whole period of preparation have been invaluable.

J. M.
J. W. R.

May, 1923.

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CHAPTER I.

NUMERICAL EVALUATION OF FORMULÆ.

I. In architecture the following rule is used for finding the depth required for a floor beam: "Take the length of the beam in feet, divide by 2 and add 2 to the answer. The result gives the depth in inches."

Calling the length of the beam l ft. and the depth d in. we may express the rule in a convenient shorthand as follows:—

$$d = \frac{l}{2} + 2.$$

Thus by simply making the first letters of the words length and depth stand for numbers, we have obtained a *formula* which expresses the rule very concisely, and from which we can calculate very easily the depth corresponding to a given length.

Ex.—*Find from the above formula the depth necessary for a beam of length* (*a*) 24 *ft.*, (*b*) 18 *ft.* 6 *in.*

The formula is $d = \frac{l}{2} + 2.$

In (*a*)

$$l = 24$$

$$\therefore d = \frac{24}{2} + 2$$

$$= 12 + 2$$

$$= 14.$$

The depth is **14 in.**

In (*b*) remembering that l is a number of feet.

$$l = 18\tfrac{1}{2}$$
$$\therefore d = \frac{18\tfrac{1}{2}}{2} + 2$$
$$= 9\tfrac{1}{4} + 2$$
$$= 11\tfrac{1}{4}.$$

The depth is $11\frac{1}{4}$ in.

Our purpose in this opening chapter is to show how algebraic formulæ in general can be evaluated when the symbols have arithmetical values assigned to them, and at the same time to explain various points in the notation employed in writing formulæ in their most concise form.

II. The Notation for Multiplication.

When we wish to represent the product of two or more algebraic symbols it is customary to omit the multiplication sign.

Thus the formula

$$A = l \times b$$

for the area of a rectangle of length l ft. and breadth b ft., is more usually written

$$A = lb.$$

In like manner

$2x$ means $2 \times x$, $\quad abc$ means $a \times b \times c$,

and so on.

The multiplication sign must of course be re-introduced when we substitute numerical values for the symbols involved. Thus the value of $5x$ when $x = 3$ is not 53, i.e. fifty-three, but 5×3 or 15.

Ex. 1.—*The volume, V cub. ft., of a room of length l ft., breadth b ft., and height h ft., is given by the formula* $V = lbh$. *Find the volume when the length, breadth, and height are respectively* 20 *ft.,* 14 *ft., and* 11 *ft.*

We have

$$V = lbh$$
$$= 20 \times 14 \times 11$$
$$= 3080.$$

The volume is **3080 cub. ft.**

Ex. 2.—*If a man walks for H hours at R miles per hour, and for h hours at r miles per hour his average speed, S miles per hour, is given by* $S = \dfrac{RH + rh}{H + h}$. *Find S when* $R = 4$, $H = 2$, $r = 3$, $h = 1\frac{1}{2}$.

$$\begin{aligned} S &= \frac{RH + rh}{H + h} \\ &= \frac{4 \times 2 + 3 \times 1\frac{1}{2}}{2 + 1\frac{1}{2}} \\ &= \frac{8 + 4\frac{1}{2}}{3\frac{1}{2}} \\ &= \frac{12\frac{1}{2}}{3\frac{1}{2}} \\ &= 3\tfrac{4}{7}. \end{aligned}$$

III. The Power Notation.

If x is any number $x \times x$ is written x^2 and is called "**the square of x**" or "**x squared.**" Similarly $x \times x \times x$ is written x^3 and is called "**x cubed**" or "**x to the power of three.**" In like manner $x \times x \times x \times x$ is written x^4 and is called "**x to the power of four,**" and so on.

Ex.—*The distance d ft. which a stone will fall in t sec. when let drop freely is given by* $d = 16t^2$. *Find how far it will fall in 3 sec.*

$$\begin{aligned} d &= 16t^2 \\ &= 16 \times 3 \times 3 \\ &= 144. \end{aligned}$$

Distance is **144 ft.**

IV. Brackets.

When brackets are used they have the same meaning as in arithmetic, viz. that the expression enclosed by them is to be treated as a single number, and should be reduced to such immediately after the numerical substitutions have been made.

Ex.—*The area of a path, x ft. wide, round the outside of a rectangle of length l ft. and breadth b ft. is given by* $A = 2x(l + b + 2x)$. *Find the area when $x = 4$, $l = 30$, $b = 20$.*

$$\begin{aligned} A &= 2x(l + b + 2x) \\ &= 2 \times 4(30 + 20 + 2 \times 4) \\ &= 8(58) \\ &= 464. \end{aligned}$$

Area = 464 sq. ft.

EXAMPLES I.

1. If $x = 3$, $y = 2$, $z = 1$, work out the values of the following :—

(*a*) $4 + x$.	(*f*) xz.	(*k*) $\frac{x}{3} + 1$.	(*p*) $11 + \frac{4}{y}$.
(*b*) $6 - y$.	(*g*) $2x + 3$.	(*l*) $2 - \frac{y}{2}$.	(*q*) $3x + 4y$.
(*c*) $y + z$.	(*h*) $8 - 4y$.	(*m*) $xy + 8$.	(*r*) $5y - 3z$.
(*d*) $4x$.	(*i*) $11 + 3z$.	(*n*) $16 - \frac{9}{x}$.	(*s*) $x + 2y + 3z$.
(*e*) $7y$.	(*j*) $10 + 5y$	(*o*) $7 - \frac{5}{z}$.	(*t*) $4y - 2x - 2z$.

2. If $a = 1$, $b = 4$, $c = 5$, find the value of :—

(*a*) $ab + c$.	(*f*) $\frac{a + b}{c}$.	(*k*) $\frac{a + 2b}{5}$.
(*b*) $a + bc$.	(*g*) $\frac{10ab}{c}$.	(*l*) $10 + \frac{12}{ab}$.
(*c*) $ac - b$.	(*h*) $bc - ac$.	(*m*) $\frac{4ab + 5}{3}$.
(*d*) $2ab + c$.	(*i*) $4ac - 5b$.	(*n*) $\frac{6ac - 5b}{2}$.
(*e*) $a + 2b - c$.	(*j*) $4abc$.	(*o*) $\frac{3abc - 8b}{7}$.

3. If $p = 6$, $q = 2$, $r = 0$, evaluate

(*a*) $p + 2q + r$.	(*f*) $\frac{3p - 4q + 5r}{5}$.
(*b*) $p - qr$.	(*g*) $\frac{2pqr}{3}$.
(*c*) $2p + 3q - 5r$.	(*h*) $\frac{p}{3} + \frac{q}{2} + \frac{r}{3}$.
(*d*) $pr - qr$.	(*i*) $\frac{r}{p} - \frac{r}{q} + \frac{p}{q}$.
(*e*) $3qr + 2pq - 4pr$.	(*j*) $\frac{7pq - 4r}{p + q}$.

4. If $y = 3x + 4$, find y corresponding to the values 0, 1, 2, 3, 4, 5 of x and tabulate your answers as below :—

x	0	1	2	3	4	5
y						

5. If $P = \frac{W}{5} + 3$, find P when W = 15.

6. The area A sq. ft. of a rectangle of length l ft. and breadth b ft. is given by the formula $A = lb$. Find the area when the length is 12 ft., and the breadth 10 ft.

7. The area A sq. in. of a triangle is given by the formula $A = \frac{1}{2}hb$ where h in. and b in. denote the height and length of base respectively. Find the area when the base is 8 in. and the height is 5 in.

8. If $v = 120 - 32t$, calculate v for the values 0, 1, 2, 3 of t and tabulate the results.

9. If $T = \frac{2x}{x + 1}$, find T for the values 0, 1, 2, 3 of x and tabulate your results.

10. If $s = a + bt$, find s when $a = 1070$, $b = 4$, $t = 15$.

11. The length l yd. of fencing required to enclose a rectangular area of 100 sq. yd., one of whose sides is x yd. long, is given by $l = 2x + \frac{200}{x}$. Make a table similar to that shown and fill in the values of l.

x	5	10	15	20	25
l					

12. The following formula occurs in mechanics : $P = \frac{2Ww}{W + w}$. Find P when W = 8, $w = 4$.

13. The volume of a box of length l ft., breadth b ft., and height h ft., is given by $V = lbh$. Find its volume in cubic feet when the length is 9 ft., the breadth 4 ft., and the height $1\frac{1}{2}$ ft.

14. Work out the values of x^2 and x^3 when x = (i) 2, (ii) 5, (iii) 7, (iv) 1.

15. If $x = 3$, $y = 4$, $z = 1$, find the value of

(*a*) x^2.	(*h*) $y^2 - z^2$.	(*o*) $3y^2 - 4x^2$.
(*b*) y^3.	(*i*) $y^3 - z^3$.	(*p*) $5z^2 + \frac{y^3}{2}$.
(*c*) z^4.	(*j*) xy^2.	(*q*) $2y^2z - 3yz^2$.
(*d*) $2x^3$.	(*k*) x^3y	(*r*) $\frac{x^4 - z^4}{10}$.
(*e*) $3y^2$.	(*l*) x^2y^2.	(*s*) $\frac{y^4 - 6}{2x^3 - 4}$.
(*f*) $4z^2$.	(*m*) xyz^3.	
(*g*) $x^2 + y^2$.	(*n*) $\frac{y^3 - x^3 - z^3}{9}$.	(*t*) $\frac{x^3z^2 - xyz}{x + y - 2z}$.

16. Fill in the blanks in the following table :—

x	0	1	2	3	4
$2x$					
x^2					
$3x$					
x^3					

17. If $s = 100 - t^2$, find s for the values 7, 8, 9, 10 of t, and tabulate the results.

18. If $T = \frac{wv^2}{g}$, find T when $w = 5$, $v = 16$, $g = 32$.

19. If $p = \frac{b}{r^2} - a$, find p when $b = 75$, $r = 5$, and $a = 2$.

20. The length of a degree of longitude in latitude $x°$ is given in miles by the formula $l = 69 - \frac{x^2}{100}$. Find the length in latitude 50°.

21. The weight of a cast-iron cube of side s in. is given in lb. by the formula $W = \frac{13}{50}s^3$. Work out the value of W when $s = 5$.

22. The area of a circle is given by the formula $A = \pi r^2$ where π may be taken to be $\frac{22}{7}$, and r in. denotes the radius of the circle. Find the area of a circle of radius $3\frac{1}{2}$ in.

23. If $f = \frac{v^2 - u^2}{2s}$, find f when $v = 30$, $u = 21$, and $s = 17$.

24. When a stone is thrown vertically upwards with a speed of 100 ft. per sec., the height h ft. attained after t sec. is given by $h = 100t - 16t^2$. Find the height reached after (i) 3 sec., (ii) 6 sec.

25. The length l ft. of wire required to stretch between two poles L ft. apart so as to permit a sag of s ft. is given by $l = L + \frac{8s^2}{3L}$. What length will be required to permit a sag of 3 ft., the distance between the poles being 50 ft. ?

26. If $s = a + bt - ct^2$, find s when $a = 10$, $b = 3$, $c = 2$, and $t = 3$.

27. If $H = \frac{a^3 + b^2c}{10}$, find H when $a = 8$, $b = 6$, $c = 3$.

28. The formula $V = \pi r^2 h$ gives the volume of a cylinder of radius r in. and height h in. Taking π to be equal to $\frac{22}{7}$, find the volume when the height is 10 in. and the radius 7 in.

29. Find the value of $(a + b)^2$ when $a = 4$ and $b = 3$.

30. By how much does $(a + b)^3$ exceed $a^3 + b^3$ when $a = 4$ and $b = 2$?

31. If the lengths of the parallel sides of a trapezium are respectively a ft. and b ft., and the perpendicular distance between them is h ft., the area A sq. ft. is given by $A = \frac{1}{2}h(a + b)$. Find the area when $a = 4\frac{1}{2}$, $b = 5\frac{1}{2}$, $h = 3$.

32. If $S = \frac{n(n + 1)(2n + 1)}{6}$, find S when $n = 0, 3, 6, 9, 12, 15$, and tabulate the results.

33. The formula $A = P\left(1 + \frac{nr}{100}\right)$ gives the amount of £P in n years at r per cent. simple interest. Find the amount when $P = 200$, $n = 4$, and $r = 5$.

34. If the top and bottom diameters of a tumbler are a in. and b in. respectively and d in. is the depth, the number of pints, P, it will hold is given by $P = \frac{d(a^2 + ab + b^2)}{132}$. Find P when $d = 6$, $a = 3$, $b = 5$.

35. The length of the hypotenuse of a right-angled triangle of sides a in. and b in. is $\sqrt{a^2 + b^2}$ in. Calculate the hypotenuse when (i) $a = 3$, $b = 4$, (ii) $a = 15$, $b = 8$, (iii) $a = 12$, $b = 35$.

CHAPTER II.

SIMPLE EXERCISES INVOLVING THE USE OF SYMBOLS.

I. We saw in last chapter that we use a letter—x for example—to represent a number whose value is not specified. If we add 3 to this number then we have $x + 3$, and can only further simplify this expression when we are told what number x stands for. Similarly, if we take 3 from x then this is expressed by writing down $x - 3$ and leaving it so, until we know the value of x or are given information sufficient to find it.

Ex. 1.—*If* $x + 3 = 7$ *what is the value of* x*?*

If $x + 3 = 7$
then since $4 + 3 = 7$
$\therefore x = 4.$

Ex. 2.—*If* $9 - w = 4$ *what is the value of* w*?*

If $9 - w = 4$
then since $9 - 5 = 4$
$\therefore w = 5.$

II. If we multiply a by b we write $a \times b$ or ab as explained in last chapter, and just as $7 \times 3 = 3 \times 7$ so $ab = ba$. If one of the factors is a known number, we place it first in the contracted way of writing the product. Thus $a \times 5 = 5 \times a = 5a$. This we can read, if we like, as five times a or five a's just as we often do in ordinary arithmetic. 5 is called the *coefficient* of a. Again $\frac{a}{6}$ is the fraction a sixths which we can also think of as $a \div 6$ or $\frac{1}{6}$ of a. In the last case it is often written as $\frac{1}{6}a$, when $\frac{1}{6}$ is called the coefficient of a.

Ex. 1.—*If* $3y + 4 = 19$ *find the value of* y.

If $3y + 4 = 19$
then since $15 + 4 = 19$
$\therefore 3y = 15.$
But $3 \times 5 = 15$
$\therefore y = 5.$

Ex. 2.—*Simplify* $\frac{a}{4} + \frac{b}{6}$.

$$\frac{a}{4} + \frac{b}{6} = \frac{3a + 2b}{12}.$$

Ex. 3.—*Simplify* $7p \times 3q$.

$$\begin{aligned} 7p \times 3q &= 7 \times p \times 3 \times q \\ &= 7 \times 3 \times p \times q \\ &= 21pq. \end{aligned}$$

EXAMPLES II (a).

Find the unknown number in each of the following examples :—

1. $x + 2 = 6.$
2. $x + 8 = 13.$
3. $s + 6 = 9.$
4. $x - 2 = 4.$
5. $z - 3 = 5.$
6. $y - 4 = 3.$
7. $3y = 12.$
8. $9z = 54.$
9. $6t = 30.$
10. $8s = 56.$
11. $\frac{x}{2} = 1.$
12. $\frac{a}{5} = 3.$
13. $\frac{y}{4} = 2.$
14. $\frac{t}{6} = 4.$
15. $\frac{m}{7} = 5.$
16. $\frac{18}{x} = 3.$
17. $\frac{20}{y} = 10.$
18. $\frac{15}{x} = 5.$
19. $\frac{28}{t} = 4.$
20. $\frac{36}{r} = 4.$
21. $2w + 9 = 15.$
22. $3y + 4 = 10.$
23. $4z + 2 = 18.$
24. $6t + 4 = 22.$
25. $4w + 9 = 25.$
26. $5y + 4 = 14.$
27. $7z + 2 = 30.$
28. $9t + 7 = 52.$
29. $3p - 6 = 3.$
30. $2v - 5 = 9.$
31. $4w - 8 = 4.$
32. $5s - 15 = 0.$
33. $12 - 2z = 10.$
34. $7 + 6t = 31.$
35. $26 - 4m = 6.$
36. $\frac{x}{2} + 3 = 5.$
37. $\frac{w}{4} - 2 = 3.$
38. $\frac{x}{3} + 4 = 6.$
39. $\frac{s}{6} - 1 = 2.$

Find a value for the unknown in each of the following examples :—

40. $y^2 = 49.$
41. $z^2 = 100.$
42. $t^2 + 2 = 18.$
43. $2z^2 = 18.$
44. $x^2 - 10 = 54.$
45. $5y^2 = 20.$
46. $10 - w^2 = 9.$
47. $t^3 = 1.$
48. $x^3 = 8.$
49. $2z^3 = 54.$
50. $\frac{s^3}{2} = 4.$

III. A statement such as 6 fives and 2 fives make 8 fives is true for every number, so that if we call the number x, then we can write this general property of numbers as follows, namely, $6x + 2x = 8x$.

Similarly, $6x - 2x = 4x$, for by taking 2 x's from 6 x's we leave 4 x's.

Thus we see that in adding or subtracting multiples of the same unknown, we apply the principles of addition and subtraction to the coefficients.

Ex.—*Simplify* $9w - 3w + 7w - 4w$.

$$\begin{aligned} 9w - 3w + 7w - 4w &= 6w + 7w - 4w \\ &= 13w - 4w \\ &= 9w. \end{aligned}$$

IV. Simplifications Involving Two Unknowns.

Ex. 1.—*A man spends x shillings on a fountain pen, y shillings on paper and envelopes, and twice as much on postage stamps as on the stationery. How much does he spend?*

He spends $(x + y + 2y)$ shillings,
that is $(x + 3y)$ shillings.

Ex. 2.—*Simplify* $9a^2 - 4a^2 + 7b^2 - 5b^2$.

$$\begin{aligned} &9a^2 - 4a^2 + 7b^2 - 5b^2 \\ &= 5a^2 + 2b^2. \end{aligned}$$

V. Miscellaneous Examples.

Ex. 1.—*An egg costs x pence. What will 12 eggs cost? What will a eggs cost? What will x eggs cost?*

1 egg costs x pence
$\therefore$ 12 eggs cost $12 \times x$ pence
i.e. $12x$ pence.

Similarly, a eggs cost ax pence,
and x eggs cost xx pence,
i.e. x^2 pence.

Ex. 2.—*If a articles cost 5 pence, what will b articles cost?*
Consider the following statement:—

If 4 articles cost 10 pence
then 1 article costs $\frac{10}{4}$ pence,

and 6 articles cost $\frac{10}{4} \times 6$ pence

i.e. $\frac{10 \times 6}{4}$ pence.

Similarly, if a articles cost 5 pence,

then 1 article costs $\frac{5}{a}$ pence

and b articles cost $\frac{5}{a} \times b$ pence,

i.e. $\frac{5 \times b}{a}$ pence

i.e. $\frac{5b}{a}$ pence.

Ex. 3.—*If x is a whole number (or integer) how will the two numbers which immediately precede it be represented?*

Consider the number 7.

The one immediately before it is $7 - 1$ or 6.

The one below this is $7 - 2$ or 5.

Similarly, if x is the number, then the two numbers immediately below it are $x - 1$ and $x - 2$.

EXAMPLES II (b).

Simplify the following expressions :—

1. $2x + 3x$.
2. $9a - 3a$.
3. $x + 5x$.
4. $7m - m$.
5. $4b + 2b - 3b$.
6. $8p - 5p + 2p$.
7. $7b - 3b - 2b$.
8. $9c - 4c - c$.
9. $5m + 3m - 7m$.
10. $7k - 4k - 2k$.
11. $9t - 7t + 5t - 2t$.
12. $11x - 8x - x + 6x$.
13. $3y^2 + 7y^2 - 4y^2 - 2y^2$.
14. $10pq - 6pq + 4pq - 7pq$.
15. $x + 2x + 4x + 5$.
16. $a + 4a + 2a - 7$.
17. $5x + 4x - 3y$.
18. $10b - 3b + 8c - 5c$.
19. $9x^2 - 2x^2 + 6y^2 - y^2$.
20. $4xy + 6xy + 9ab - 5ba$.
21. A bricklayer lays b bricks in 1 day. How many will he lay in (i) 4 days; (ii) x days; (iii) b days?
22. A bottle of port costs x/-. What will 1 doz. bottles cost?
23. Reduce £p to shillings; y yd. to feet; g gal. to pints.
24. Express d pence in shillings; c cwt. in tons; m min. in hours.
25. If $2x + 7x = 18$, find the value of x.

26. If $9t - 4t = 15$, find the value of $2t$.

27. $2x$ golf balls are bought at 3/-. each. Find their cost.

28. A line $10l$ in. long is divided into 5 equal parts. Find the length of each part.

29. A man buys 3 articles at d pence each. Find their cost (i) in pence; (ii) in shillings.

30. 1 lb. of tobacco costs s shillings. How much is this per oz.? Give the result (i) in shillings; (ii) in pence.

31. If $\frac{6}{x} + 4 = 7$, find x.

32. A box contains 10 packages each weighing x lb. If the box itself weighs 7 lb., find the total weight.

33. 40 doz. eggs are bought at s/- per doz. In addition the purchaser pays 2/- for delivery. What does he pay in all?

34. If 2 articles cost xd. what will (i) 5 articles cost; (ii) a articles cost?

35. The prices of admission to a concert are respectively. front seats 2/- each, back seats 1/- each. There are x people in the front seats and $3x$ people in the back seats. Find the total drawings in £.

36. A reel of thread is $20x$ yd. long. From it seven lengths of $2x$ yd. are cut off. How much remains.

37. Simplify:—

(i) $2x \times 3y$. (ii) $4a \times 5a$. (iii) $\frac{a}{3} \times \frac{b}{5}$.

(iv) $\frac{b}{a} \times \frac{3}{c}$. (v) $5 \times 3x$. (vi) $5x \times 3$.

(vii) $4a \times 6$. (viii) $3 \times 2b + 4 \times 6b$. (ix) $6 \times 7y - 8y \times 3$.

(x) $6 \times \frac{c}{3}$. (xi) $\frac{m}{8} \times 4$. (xii) $\frac{9}{z} \times \frac{1}{6}$.

38. n is a whole number. What is the next higher number? What is the sum of these two numbers?

39. A boy is 10 yr. old. Find his age (i) x yr. hence; (ii) x yr. ago.

40. A cwt. of sugar is bought for x/- and sold at $\frac{x}{2}$d. per lb. Find the profit in pence.

41. Simplify:—

(i) $\frac{x}{2} + \frac{x}{4}$. (ii) $\frac{w}{3} + \frac{w}{6}$. (iii) $\frac{m}{4} - \frac{m}{6}$. (iv) $\frac{z}{3} - \frac{z}{4}$.

42. Simplify:—

(i) $\frac{a}{2} + \frac{b}{3}$. (ii) $\frac{b}{3} + \frac{c}{4}$. (iii) $\frac{f}{4} - \frac{g}{6}$. (iv) $\frac{g}{6} - \frac{h}{9}$.

43. A page of a book is x in. wide and $2x$ in. deep. Find its perimeter. Verify your answer when the page is 4 in. wide.

CHAPTER III.

DIRECTED NUMBERS. THEIR ADDITION AND SUBTRACTION.

I. In arithmetic the signs + and − are employed to indicate the two simple operations of addition and subtraction. We shall now explain their further use in algebra.

In our calculations we are often concerned with something more than the mere number of units in question. We require to know whether we are reckoning up or down, right or left, East or West, before or after, an increase or a decrease, etc. This is accomplished in algebra by prefixing the signs + and − to numbers, on the understanding that if by + £6 we mean a gain of £6 then by − £6 we mean a loss of £6, or if by + 5 ft. we mean 5 ft. reckoned upwards then by − 5 ft. we mean 5 ft. reckoned downwards, or again if + 3 hr. stand for 3 hr. after noon then − 3 hr. will stand for 3 hr. before noon, and so on.

As a practical illustration of the use of the signs + and − as described above, we may instance the measurement of temperature by the centigrade thermometer where temperatures below the freezing-point of water are marked −, while those above have no sign attached, but are understood to carry the sign +. Thus by a temperature of − 6° we mean a temperature 6° below freezing-point (i.e. zero), while a temperature of 8° means a temperature 8° above zero.

Numbers such as + 8 and − 6 are called "**Directed Numbers,**" the first type being called **Positive Numbers** and the second type **Negative Numbers.** It is usual, just as in the case of temperatures, to leave the positive numbers unmarked. The point from which reckonings are made is called the zero point.

In this chapter we shall study how directed numbers are "added" and "subtracted."

Ex. 1.—*Translate the following expressions :—*

The post is – 7 minutes early ; my weight has decreased by – 10 lb. ; I banked – £5 ; the train went on – 6 miles.

(i) The post is 7 minutes late. (ii) My weight has increased by 10 lb. (iii) I withdrew £5 from the bank. (iv) The train went back 6 miles.

Ex. 2.—*In a game of golf express* 3 *holes lost as holes won ; express* 60 *ft. upwards as feet downwards ; if a boat discharges* 18 *tons of cargo and takes on* 22 *tons express the change in the weight of the cargo as a decrease.*

(i) – 3 holes won. (ii) – 60 ft. downwards. (iii) – 4 tons less of cargo.

EXAMPLE III (a).

1. Translate the following expressions :—

– 5 ft. above sea level ; – £8 profit ; – £10 loss ; – 6 miles North ; – 7 miles West ; a spring is extended – $1\frac{1}{2}$ in. ; a rise in temperature of – 14° ; a man's assets are – £60 ; a patient gains – $2\frac{1}{2}$ lb. in weight ; – $3\frac{1}{4}$ gal. of water flow into a tank ; a patient's temperature falls – 2° ; the cost of living falls – 6 points ; an army retreats – 6 miles ; a clock is – 4 minutes slow ; the population of a town increases by – 2000.

2. Express a loss of £7 as a gain ; express a rise of temperature of 3° as a fall ; express 8 miles South as miles North ; if a man buys an article for £6 and sells it for £4 find his profit.

3. Complete the following table which shows the number of deaths from influenza in various cities during 1918 for the two weeks mentioned :—

City.	Week ending 9th Nov.	Week ending 16th Nov.	Increase.
Birmingham	212	246	
Bradford	123	122	
Bristol	183	144	
Leeds	253	208	
Manchester	159	235	
Nottingham	189	198	
Sheffield	318	241	

4. At noon four clocks indicate the following times, namely, 11.55 A.M.; 12.7 P.M.; 12.11 P.M.; 11.48 A.M. State the amount each is fast or slow, using + to indicate "fast" and − to indicate "slow."

5. A body is at 15° C. and is being cooled. State its temperature after it has fallen 5°, 10°, 15°, 20°, 25°.

6. In an experiment a substance was cooled to a temperature of − 12°. As it returned to room conditions at the end of the experiment, state its temperature after it has risen 4°, 8°, 12°, 18°, 27°.

7. Proof whisky contains 57 per cent. of alcohol. Four samples were found to contain 35 per cent., 67 per cent., 86 per cent., and 40 per cent. of alcohol respectively. State the variations from proof in each case, using + to indicate over proof and − to indicate under proof.

8. A pit head is 100 ft. above sea level. A cage descends the shaft at the rate of 5 ft. per second. State the position of the cage relative to sea level after 5, 15, 25, 35, and 45 seconds, using the signs + and −.

9. A goods train is travelling Northwards towards a certain station from which it is 70 miles distant at the rate of 30 miles per hour. State its position relative to the station after 1 hr., 2 hr., 3 hr., 4 hr., using + to indicate North and − to indicate South of the station.

II. Algebraic Addition.

The meaning of this process is best explained by examples.

If, for instance, a train proceeds 20 miles and then goes back 8 miles, what is the net result? The obvious answer is, that the train has progressed just 12 miles. Put algebraically, this means that + 20 and − 8 make + 12.

Again, suppose that a thermometer rises 4° and then falls 9°, what is the net result? The two movements are equivalent to a simple fall of 5°. In algebraic terms this means that + 4 and − 9 make − 5.

Another illustration would be the case of a banker who begins the day's business with, say, £4000, and proceeds to make the following transactions, namely: pays out, £10; takes in, £6; takes in, £9; pays out, £11. What is the net effect of these transactions? They are obviously equivalent to a simple "pay out" of £6. Thus − 10 and + 6 and + 9 and − 11 make − 6, the sign + marking money received and the sign − money paid out.

This finding of the net result is what is called "taking the algebraic sum of the directed numbers."

Taking the last example, it is easy to show that the sum can be effected in various ways, as it obviously does not matter in what order the transactions arise. A common practice is, first to find how much is "taken in" then how much is "paid out," and finally to balance the two results. In algebraic language this means that we add all the positive numbers, then all the negative ones, and finally set the one result against the other to obtain the "net result" or "sum." It is to be noted that $+ x$ and $- x$ when added give a zero result, since, for example, if a man receives £x and then pays them away he leaves off just where he began.

Ex. 1.—*Find the sum of the following numbers :—*

$+ 6, - 9, - 3, + 12, - 2.$

Sum of positive numbers $= + 18$
Sum of negative numbers $= - 14$
Net result or sum $+ 4$.

Ex. 2.—*Find the sum of* $- 7x, - 2x, 8x, - 5x$ and $3x$.

N.B.—As already stated, when the sign of a number is not given it is understood to be $+$.

Sum of positive numbers $= 11x$
Sum of negative numbers $= - 14x$
Net result or sum $= - 3x$.

EXAMPLES III (b).

Find the sum of :—

1. 5 and $- 2$.
2. $-$ 9 and 4.
3. $- 7$ and $- 8$.
4. $6x$ and $- 4x$.
5. $- 5y$ and $+ 2y$.
6. $- 4y$ and $- 7y$.
7. $6w$, $5w$ and $- 9w$.
8. $- 2t$, $6t$ and $- 8t$.
9. $4m$, $3m$ and $- 9m$.
10. $- 5s$, $- 8s$, and $- 9s$.
11. $5a$, $- 3a$, $7a$ and $- 6a$.
12. $- v$, $- 2v$, $5v$ and $8v$.
13. $- 4x$, $- 6x$, $7x$ and $- 11x$.
14. $- 3z$, z, $- 6z$ and $8z$.
15. $2c$, $- 9c$, $- 5c$ and $8c$.
16. $- 4x^2$, $6x^2$, $- 7x^2$ and $9x^2$.
17. $3xy$, $- 6xy$, $- 8xy$ and $7xy$.
18. $7pq$, pq, $- 6pq$ and $- 2pq$.
19. a^2, $- 2a^2$, $5a^2$, $- 7a^2$ and $8a^2$.
20. $- xyz$, $3xyz$, $- 5xyz$, $7xyz$, and $- 9xyz$.

III. Algebraic Subtraction.

When we subtract one number from another in arithmetic, we resolve the primary number into two parts, one of which is the subtracted number and the other is what we call the "remainder". This process is continued in algebra.

Ex. 1.—*From* 3 *subtract* − 7.
The result when added to − 7 must give 3.
Now 7 added to − 7 gives 0.
∴ add 3 more and we shall have 3.
Thus in all we add 7 and 3 or **10.**
Check: 10 and − 7 make 3.

Note that the remainder 10 is found by changing the sign of the subtracted number (− 7) and adding the result (7) to 3.

By selecting any pair of directed numbers and proceeding as above we can establish the following working rule, namely, "to find the difference between two directed numbers, change the sign of the subtracted number and add the result to the other given number."

Ex. 2.—*Subtract* $4x$ *from* $-11x$.
Changing the sign of the subtracted number gives $-4x$.
Adding $-4x$ to $-11x$ gives $-15x$.
Ans. $-15x$.
Check: $-15x$ and $4x$ give $-11x$ when added.

EXAMPLES III (c).

From

1. 6 subtract − 2.
2. − 7 „ 3.
3. 5 „ 9.
4. − 4 „ 7.
5. − 3 „ − 8.
6. 10 „ − 6.
7. $7x$ subtract $2x$.
8. $3a$ „ $8a$.
9. $-6c$ „ $4c$.
10. $5c^2$ „ $-7c^2$.
11. $-3xy$ „ $8xy$.
12. $9pq$ „ $-6pq$.

Subtract

13. − 6 from 9.
14. 8 „ 2.
15. − 7 „ − 3.
16. a „ $6a$.
17. $6a$ „ a.
18. $7x^2$ from $-3x^2$.
19. $-2ab$ „ $-7ab$.
20. $6pq$ „ $-4pq$.
21. $-7c^2$ „ $4c^2$.

22. The mean summer temperature of a place is 63° F. and the mean winter temperature is − 15° F. Find the range.

23. At the end of an experiment the temperature of a substance rose from − 20° C. to 15° C. Find the rise in temperature?

24. At the beginning of a year a man's assets are − £200 and at the end of the year they are £100. What profit has he made during the year?

IV. Simplification of Expressions.

Ex. 1.—*Simplify* $4 - 9 - 6 + 7$.

Proceeding step by step we first subtract 9 from 4, then subtract 6 from the result and, finally, add 7 to the last result.

$$\begin{aligned} \therefore 4 - 9 - 6 + 7 &= -5 - 6 + 7. \\ &= -11 + 7. \\ &= -4. \end{aligned}$$

This answer is the same as that which we get by summing the numbers 4, − 9, − 6, 7.

Terms: The numbers 4, − 9, − 6, and 7 are called the terms of the expression $4 - 9 - 6 + 7$.

The usage applies to all similar expressions.

Thus in the expression $3x - 4y + 7x - 9y - 2y$ the terms are $3x$, $-4y$, $7x$, $-9y$ and $-2y$.

Ex. 2.—*Simplify* $5x - 2x - 6x + 9x - x$.

Proceeding step by step we have

$$\begin{aligned} 5x - 2x - 6x + 9x - x &= 3x - 6x + 9x - x \\ &= -3x + 9x - x \\ &= 6x - x \\ &= 5x. \end{aligned}$$

But this result can also be obtained by effecting the sum of the numbers $5x$, $-2x$, $-6x$, $9x$ and $-x$. This can be done by summing the positive and negative numbers separately and then adding these results. Hence we can sum the positive and negative terms of the expression separately and then simplify the result.

$$\begin{aligned} \therefore 5x - 2x - 6x + 9x - x &= 14x - 9x \\ &= 5x. \end{aligned}$$

This procedure is often followed.

EXAMPLES III (d).

Simplify the following expressions :—

1. $5 - 2$.
2. $2 - 5$.
3. $6 - 4$.
4. $4 - 6$.
5. $7 - 3$.
6. $3 - 7$.
7. $-7 + 3$.
8. $-7 - 3$.
9. $-4 + 9$.
10. $-5 - 8$.
11. $9x - 5x$.
12. $4a - 7a$.
13. $5x - 5x$.
14. $-7y + 7y$.
15. $-3w + 7w - 2w$.
16. $7a - 3a - 8a$.
17. $-a - 11a + 3a$.
18. $3p + 7p - 6p$.
19. $-7b - 5b + 3b$.
20. $-8c + 6c - 7c$.
21. $6x - 9x + 8x - 2x$.
22. $-8y + 2y - 7y + 6y$.
23. $7z - 3z - 9z + 6z$.
24. $-3t + 5t + 4t - 11t$.
25. $-9s - 3s + 7s - 2s$.
26. $-7c - 6c + 8c + 2c$.
27. $c - 9c - c + 2c$.
28. $-9m - 3m + 7m - 2m + 4m$.
29. $5q - 7q - 8q + 9q - 12q$.
30. $b - 7b - 6b + 8b - 5b + 9b$.

V. Harder Additions.

Ex.—*Find the sum of* $5x - 3y$; $-2x + 4y$; $-x + 5y$; $4x - 9y$.

The answer must be an expression giving the total of the x terms and of the y terms separately. For convenience we shall place the expressions one below the other in a manner similar to the plan followed in arithmetic.

$$\begin{array}{rr} & 5x - 3y \\ & -2x + 4y \\ & -x + 5y \\ & 4x - 9y \\ \hline \text{Total} & 6x - 3y \end{array}$$

EXAMPLES III (e).

Find the sum of :—

1. $3x - 7y$; $-2x + 8y$.
2. $-9a - 7b$; $a + 3b$.
3. $-7r + 2t$; $9r - 4t$.
4. $-s + 6w$; $10s - w$.
5. $6b - c$; $2b + 9c$.
6. $-7a - 2b$; $a + 6b$.
7. $-3p + 8q$; $7p - 9q$; $p + 7q$.
8. $-3y + 4z$; $7y - 2z$; $9y + 8z$.
9. $4x - 2y$; $-5x + y$; $7x - 9y$.

10. $-s+3t$; $s-7t$; $-8s-4t$.
11. $5v^2-4w^2$; $-9v^2-5w^2$; v^2+7w^2.
12. $-xy+6ab$; $5xy-3ab$; $-7xy-4ab$.
13. $-4a+2b-5c$; $9a-7b+6c$.
14. $2x+4y-8z$; $-5x-9y+7z$.
15. $6x^2-5y^2-10a^2$; $-9x^2+12y^2-8a^2$; $5x^2-7y^2+4a^2$.
16. $5p^2+2q^2-6r^2$; $-6q^2+4r^2$; $-3p^2+5r^2$.
17. x^2+4x-6; $5x^2-7x+2$; $-9x^2+6x+7$.
18. $y^2+10y-8$; $8y^2-6$; $-5y+7$.
19. $8t^2-t+2$; $-6t^2+3t$; $-9t+5$.
20. $2a-3b$; $-7a+5b$; $8a-4b$; $-a-5b$; $3a-b$.
21. $6x+3y$; $2y-7x$; $9y-5x$; $-4x-2y$; $8x+y$.
22. $5l-3m$; $8m-2l$; $-6l-2m$; $2l+4m$; $l+m$.
23. $5y^2-3x^2$; $6x^2-8y^2$; x^2+7y^2; $-4x^2-y^2$; $4y^2-x^2$.
24. $4a^2-3xy$; $6xy-5a^2$; $7a^2+9xy$; $-5xy$; $-9a^2$.

VI. Harder Subtractions.

Ex.—*Subtract* $9x-8y$ *from* $3x-5y$.

$$\begin{array}{r} 3x-5y \\ 9x-8y \\ \hline -6x+3y \end{array}$$ Ans.

The answer is obtained by adding $-9x$ to $3x$ and $+8y$ to $-5y$ in accordance with the rule, "change the sign of the subtracted number and add it to the other." The correctness of the answer can be verified, by adding it to the second line.

EXAMPLES III (f).

Subtract :—

1. $6a+2b$ from $4a-3b$.
2. $-3a-5b$ from $6a-2b$.
3. $7x^2-3y^2$ from $4x^2-9y^2$.
4. $x+7y$ from $3x-4y$.
5. $-8c-d$ from $-2c+3d$.
6. $-7a^2+2b^2$ from $5a^2-3b^2$.
7. $-2x^2+9ab$ from $-6x^2+3ab$
8. $-6x^2-7y^2$ from x^2-y^2.
9. $4pq+5rs$ from $-7pq-3rs$.
10. $5ab-7c^2$ from $-5ab+7c^2$.
11. $7t^2-4$ from $2t^2+6$.
12. $-8bc+a^2$ from $8bc-a^2$.
13. $3x^2+7x-2$ from $8x^2-2x+5$.
14. $-9y^2+8y+10$ from $-2y^2+6y-4$.
15. $-5a+3b+6c$ from $a+2b-9c$.
16. $3p^2-q^2+5r^2$ from $p^2+6q^2+5r^2$.
17. $2a+3b$ from $6a-9b-2c$.
18. $-p+6r$ from $5p-2q-3r$.

19. $4y - 5z$ from $-3x - y - 9z$.
20. $2x - 3y + 4z$ from $5x - 6z$.
21. $3a^2 - 4b^2 - 9c^2$ from $6b^2 - 2c^2$.
22. $3y^2 - y + 3$ from $y^2 - 4$.
23. $-y^2 + 7y - 4$ from $2y - 6$.
24. $-4y^2 - y - 6$ from $y^2 - y$.
25. By how much does $3x - 2$ exceed $2x - 5$.
26. From the sum of $a + 3b$ and $-6a - 2b$ take $4a - b$.
27. From $3x - 7y$ take the sum of $2x - 8y$ and $-6x + 3y$.
28. What must we add to $6p - 2q$ to obtain $-3p + 8q$?
29. What must be subtracted from $6s - 5t$ to obtain $2s + 3t$?
30. From $a - 3b - 7c$ take $4a + 2b - 5c$ and to the result add $3a - b - 6c$.
31. By how much does the sum of $-2x + 4y - z$ and $5x - 9y + 7z$ exceed the sum of $6x - 4y$ and $7y - 3z$?
32. Subtract $3x^2 - 5y^2 - z^2$ from zero.

CHAPTER IV.

MULTIPLICATION AND DIVISION OF DIRECTED NUMBERS AND THEIR POWERS.

I. We already know that 2×3 (or three times 2) is 6, being equal to $2 + 2 + 2$.

There will be no difficulty in seeing that -2×3 (or three times -2) is -6 being equal to $-2-2-2$.

When, however, we come to deal with examples like 2×-3 and -2×-3 where the multiplier -3 is a negative number, we must return to a study of the construction of our multiplication tables to see how their principles may be extended to cover such cases.

−3	**−2**	**−1**	**0**	**1**	**2**	**3**	**4**	
−9	−6	−3	0	3	6	9	12	**3**
−6	−4	−2	0	2	4	6	8	**2**
−3	−2	−1	0	1	2	3	4	**1**
0	0	0	0	0	0	0	0	**0**
3	2	1	0	−1	−2	−3	−4	**−1**
6	4	2	0	−2	−4	−6	−8	**−2**
9	6	3	0	−3	−6	−9	−12	**−3**

In the above diagram the numbers in the top row (heavy type), namely $-3, -2, -1, 0, 1, 2, 3, 4$, are multiplied by the numbers in the outside right-hand column (heavy type) in turn, that is to say by $3, 2, 1, 0, -1, -2, -3$, and the results are tabulated in the other rows. Thus when we multiply the top row by 3 we get the first row of figures, namely $-9, -6, -3, 0, 3, 6, 9, 12$, since three times

-3 is -9, three times -2 is -6 and so on. We can perform the multiplications as far as the 0-times line, but as yet do not know how to go on.

If, now, we examine the results in the column headed -3 (heavy type), we see that each figure in it is obtained by subtracting -3 from the one above it. Hence, in order that our new products may be derived by the continuing of this process we must subtract -3 from zero, then again from this result and so on. Thus after reaching zero the table will go on 3, 6, 9, etc. If we are to extend the next column we must evidently proceed by subtractions of -2 so that below zero we shall place 2, 4, 6, etc. In this way the chart can be completed. Thus, we see that, admitting this continuity of process, we must have

(i)
$$\begin{aligned} -3 \times -1 &= 3 \\ -2 \times -1 &= 2 \\ -1 \times -1 &= 1 \\ 0 \times -1 &= 0 \\ 1 \times -1 &= -1 \\ 2 \times -1 &= -2 \\ 3 \times -1 &= -3 \\ 4 \times -1 &= -4 \end{aligned}$$

(ii)
$$\begin{aligned} -3 \times -2 &= 6 \\ -2 \times -2 &= 4 \\ -1 \times -2 &= 2 \\ 0 \times -2 &= 0 \\ 1 \times -2 &= -2 \\ 2 \times -2 &= -4 \\ 3 \times -2 &= -6 \\ 4 \times -2 &= -8 \end{aligned}$$

and so on. We also find that $-2 \times -3 = 2 \times 3 = 6$ and that $2 \times -3 = -2 \times 3 = -6$, or in more general terms that if a and b be any two arithmetical numbers then

$$-a \times -b = a \times b = ab \text{ and } -a \times b = a \times -b = -ab.$$

These most important principles may be enunciated as follows:—

> **"If two directed numbers be multiplied together, then if they have like signs their product is a positive number, and if they have unlike signs their product is a negative number."**

This statement is often put in the following brief form namely, that in the multiplication of two directed numbers,

> **"like signs give + and unlike signs give − ."**

It remains to note that the tables show us that

$$2 \times -3 = -3 \times 2,$$

or that $ab = ba$ as in arithmetic.

Ex. 1.—*Find the value of* -6×9.

The signs of the factors being unlike the product will be a negative number, and since $6 \times 9 = 54$,

$$\therefore -6 \times 9 = -54.$$

Ex. 2.—*Find the value of* $-3 \times 5 \times -6$

$$-3 \times 5 \times -6 = -15 \times -6$$
$$= 90.$$

Ex. 3.—*Find the value of* $4 - 3x$ *and of* $5 - x$ *when* $x = -2$.

(i) When $x = -2$ then $4 - 3x = 4 - 3 \times -2$, that is to say we have to subtract 3×-2 or -6 from 4. But subtracting -6 from 4 is equivalent to adding 6 to 4

$$\therefore 4 - 3x = 4 - 3 \times -2$$
$$= 4 + 6$$
$$= 10.$$

(ii) When $x = -2$ then

$$5 - x = 5 - 1x$$
$$= 5 - 1 \times -2$$
$$= 5 + 2 \text{ (reasoning as in i)}$$
$$= 7.$$

EXAMPLES IV (a).

Find the value of

1. -7×2; -6×5; -8×4.
2. 9×-3; 8×-7; 6×-11.
3. -2×-5; -4×-6; -7×-9.
4. 5×-8; -9×-3; -8×7.
5. -6×-7; -8×-3; 7×-7.
6. -5×5; 8×-8; -12×-12.
7. 4×-5; -8×-9; -13×-2.
8. $4 \times -5 \times 7$.
9. $7 \times 4 \times -2$.
10. $-2 \times -5 \times -8$.
11. $-3 \times -4 \times 9$.
12. $-4 \times 6 \times -8$.
13. $2 \times -3 \times 4 \times -5$.
14. $-2 \times -3 \times 4 \times -5$.
15. $-2 \times -3 \times -4 \times -5$.
16. $-2 \times 3 \times -4 \times -5$.
17. If $P = mf$ find the value of P when $m = 6$ and $f = -2$.
18. If $z = xy$ find z when $x = -4$ and $y = -8$.

19. If $n = 5lm$ find n when $l = -3$ and $m = -6$.
20. If $c = 4ab$ find c when $a = -2$ and $b = 11$.
21. If $w = \frac{1}{2}pq$ find w when $p = 8$ and $q = -4$.
22. If $f = \frac{3}{4}gh$ find f when $g = -10$ and $h = -6$.
23. Find the value of x^2 when $x = 2, -2, 3, -3, 5, -5, 7, -7, \frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}$.
24. Find the value of m^3 when $m = -2, -4, -6, -8, -\frac{1}{3}, -\frac{1}{5}$, and $-\frac{1}{7}$.
25. Find the value of at^2 when $a = 3$ and $t = -4$.
26. Find the value of ct^3 when $c = 6$ and $t = -2$.
27. Find the value of x^2y^3 when $x = -8$ and $y = -4$.
28. Find the value of r^4 when $r = 2, -3, -4$, and 5.
29. Find the value of $2x^4 - 7x^2 + 1$ when $x = -3$.
30. Find the value of $4 - 3x$ when $x = -3, -2, 4$, and 5.
31. Find the value of $-5y + 6$ when $y = -8, -3, 2$, and 5.
32. Find the value of $6 - x$ when $x = -9, -4, 6$, and 8.
33. If $y = 5 - 3x$ complete the following chart:—

x	-3	-2	-1	0	1	2	3
y							

34. If $v = -10 - 6t$ complete the following chart:—

t	-5	-3	-1	0	2	4
v						

35. If $y^2 = 25$ prove that y may be 5 or -5.

II. Division of Directed Numbers.

A few examples will show that in the division of two directed numbers "**like signs give + and unlike signs give −**" just as in multiplication.

Ex. 1.—*Find the value of* $\frac{21}{-7}$.

The quotient is such that when multiplied by -7 the result must be 21.

But $-7 \times -3 = 21$.

$$\therefore \frac{21}{-7} = -3.$$

Ex. 2. *Find the value of* $\frac{-30}{-6}$.

The quotient when multiplied by -6 must give -30.

But $-6 \times 5 = -30$.

$$\therefore \frac{-30}{-6} = 5.$$

EXAMPLES IV (b).

Find the value of

1. $\frac{-8}{2}$; $\frac{6}{-3}$; $\frac{-12}{-4}$.
2. $\frac{42}{-6}$; $\frac{-48}{-8}$; $\frac{-40}{5}$.
3. $\frac{-35}{-7}$; $\frac{-36}{9}$; $\frac{20}{-4}$.
4. $\frac{-30}{2}$; $\frac{-45}{-5}$; $\frac{56}{-8}$.
5. $\frac{54}{-6}$; $\frac{-42}{3}$; $\frac{-63}{-7}$.
6. $\frac{-52}{-4}$; $\frac{99}{-11}$; $\frac{-84}{12}$.
7. Find the value of $\frac{y}{z}$ when $y = 12$ and $z = -3$.
8. Find the value of $\frac{4l}{m}$ when $l = -9$ and $m = 6$.
9. Find the value of $\frac{s}{3t}$ when $s = -72$ and $t = -4$.
10. Find the value of $\frac{xy}{z}$ when $x = -18$, $y = 3$ and $z = -6$.
11. If $e = \frac{3fg}{4hk}$ find e when $f = -9$, $g = 8$, $h = 6$ and $k = -2$.
12. If $v = \frac{2s}{t^2}$ find v when $s = -50$ and $t = 5$.
13. Find the value of $\frac{v^2}{2f}$ when $v = -12$ and $f = -4$.
14. Find the value of $\frac{xy^3}{z^2}$ when $x = -6$, $y = -2$ and $z = -4$.

III. Products Involving Factors of Unknown Value.

Ex. 1.—*Find the value of* $5a \times 4b$.

$$5a \times 4b = 5 \times a \times 4 \times b$$
$$= 20\,ab.$$

Ex. 2.—*Find the value of* $2x \times -5y \times -z$.

$$2x \times -5y \times -z = 10xyz.$$

We may carry out the working in two steps thus:—

$$2x \times -5y \times -z = -10xy \times -z = 10xyz.$$

IV. Products involving Powers of the Same Number.

We must first recall that x^4 is read x to the power 4 and means $x \times x \times x \times x$; 4 is called the **index** of the power, and shows we are dealing with the product of 4 factors x. Other powers are defined in the same way.

Ex. 1.—*Find the value of* $x^2 \times x^3$.

Since $x^2 = x \times x$ and $x^3 = x \times x \times x$

$\therefore x^2 \times x^3 = x \times x \times x \times x \times x = x^5$. (Note that $5 = 2 + 3$.)

The index of the product is evidently the sum of the given indices.

Ex. 2.—*Find the value of* $t^2 \times t^4 \times t^9$.

$$t^2 \times t^4 \times t^9 = t^6 \times t^9$$
$$= t^{15}.$$

(Note that $15 = 2 + 4 + 9$.)

Ex. 3.—*Find the value of* $p \times -3p^2 \times 2p^5$.

$$p \times -3p^2 \times 2p^5 = p \times -3 \times p^2 \times 2 \times p^5$$
$$= -6 \times p \times p^2 \times p^5$$
$$= -6p^8. \quad (N.B. \quad p = p^1).$$

Ex. 4.—*Find the value of* $x^2y^3 \times ax^4y^5$.

$$x^2y^3 \times ax^4y^5 = x^2 \times y^3 \times a \times x^4 \times y^5$$
$$= a \times x^2 \times x^4 \times y^3 \times y^5$$
$$= ax^6y^8.$$

V. Quotients Involving Powers of the Same Number.

Ex. 1.—*Find the value of* $\frac{x^5}{x^2}$.

Since $x^5 = x^2 \times x^3$

$$\therefore \frac{x^5}{x^2} = \frac{\cancel{x^2} \times x^3}{\cancel{x^2}} = x^3.$$

The index of the result is evidently the difference of the given indices.

Ex. 2.—Simplify $\frac{-15ab^6c^8}{-5ab^2c^4}$.

$$\frac{-15ab^6c^9}{-5ab^2c^4} = 3b^4c^5 \left(N.B. \quad \frac{a}{a} = 1\right).$$

EXAMPLES IV (c).

Simplify the following products :—

1. $5a \times 2b$; $3b \times 7c$; $4p \times 3q$; $6x \times 9y$; $2a \times 3b \times 4c$; $f \times 4g \times 5h$; $3p \times q \times 6r$; $2x \times 9y \times z$.
2. $-3a \times -8b$; $5x \times -4y$; $-9s \times 3t$; $2p \times -4q \times 5r$; $-3l \times 5m \times -6n$; $4a \times -5b \times -c$; $-2f \times -3g \times -7c$; $-4x \times -y \times 8z$; $-5r \times 6s \times t$.
3. $x^2 \times x^3$; $x^2 \times x^6$; $x^3 \times x^4$.
4. $a^2 \times a^5$; $a^3 \times a^7$; $a^4 \times a^8$.
5. $v \times v^4$; $v^9 \times v$; $v \times v^2 \times v^3$.
6. $y^2 \times y^3 \times y^5$; $y^4 \times y^2 \times y^6$; $y^7 \times y^3 \times y$.
7. $x^3 \times -x^5$; $-x^4 \times -x^6$; $-x^8 \times x^2$.
8. $-m^2 \times m^3 \times -m^5$; $m^4 \times -m^2 \times -m^6$.
9. $-z^2 \times z^4 \times z \times -z^9$.
10. $2x \times 3x^2$; $4x^2 \times 5x^3$.
11. $3y^2 \times 7y^3$; $6y^4 \times 5y^2$.
12. $-8b^3 \times 4b^2$; $-6b^4 \times -5b^6$.
13. $-9s^5 \times -2s^4$; $7s^8 \times -4s$.
14. $-6f^6 \times -2f \times -5f^8$.
15. $2x^4y^2 \times 7x^3y^5$; $-4x^2y^6 \times -9x^5y^7$.
16. $3a^2b \times 4a^3$; $2ab^2 \times 5a^3b$.
17. $4a^2 \times -5a \times 2b$.
18. $-5 \times -3a^2b \times 2ab$.
19. $-3a \times -4b \times -2ab$.
20. $4ac \times -8ad \times 2cd$.
21. $-2a^3b \times -3ab^3 \times -3a$.

Find the quotient in each of the following cases :—

22. $\frac{x^7}{x^4}$; $\frac{x^9}{x^5}$; $\frac{x^{11}}{x^3}$.
23. $\frac{-y^6}{y^2}$; $\frac{-y^9}{y^4}$; $\frac{-y^{12}}{y^3}$.
24. $\frac{z^7}{-z^4}$; $\frac{z^{10}}{-z^6}$; $\frac{z^9}{-z}$.
25. $\frac{-a^8}{-a^4}$; $\frac{-a^9}{-a}$; $\frac{-a^{16}}{-a^{10}}$.
26. $\frac{2t^8}{t^4}$; $\frac{6t^8}{3t^7}$; $\frac{20t^9}{4t^5}$.
27. $\frac{9s^7}{-3s^2}$; $\frac{-8s^9}{4s^3}$; $\frac{-12s^{11}}{-6s^5}$.
28. $\frac{a^2b}{ab}$; $\frac{a^3b^4}{ab^2}$; $\frac{6a^4b^5}{2a^3b}$.
29. $\frac{15b^5c^6}{-5bc}$; $\frac{-20b^4c^6}{-5b^2c^3}$; $\frac{-36b^8c^9}{4b^7c}$.
30. $\frac{16b^2c}{-8bc}$; $\frac{-30b^4c}{-6b^3c}$; $\frac{-42b^2c^8}{7b^2c^5}$.
31. $\frac{b^2}{b^5}$; $\frac{2a^3b^4}{8a^5b}$; $\frac{21ab^2c}{7a^3b^5c}$.

CHAPTER V.

BRACKETS.

I. The Use of Brackets.

We already know something of the use of brackets, having met them in the examples of Chapter I., as for instance in the formula $A = \frac{1}{2}h(a + b)$.

Instead of using brackets of the type ( ) we may use a bar or vinculum ———, drawn above the terms; thus

$$a + (b + c) = a + \overline{b + c}.$$

When more complicated expressions arise the following forms of brackets are used { } and [ ], as for example

$$\{(5x + 2y) + (3x - 4y)\} + \{(7x - y) - (5x + 6y)\}.$$

Here we have to simplify the expressions within the brackets, { }, and then add the results together.

II. The Type $a(x - y + z)$. Removal of Brackets.

If we take the expression $3(x - y + z)$, or three times $x - y + z$ then we know that we have to deal with the sum of three expressions, each $x - y + z$. Perform their addition

$$\begin{array}{r} x - y + z \\ x - y + z \\ x - y + z \\ \hline 3x - 3y + 3z \end{array}$$

Hence $3(x - y + z) = 3x - 3y + 3z$. The terms on the right-hand side are evidently obtained by multiplying by 3 each of the numbers x, $-y$ and z. In general then

$$a(x - y + z) = ax - ay + az.$$

Ex. 1.—*Simplify* $-5(3x - 2y)$.

Multiplying each of the quantities $3x$ and $-2y$ by -5 gives us $-5(3x - 2y) = -15x + 10y$.

Ex. 2.—*Simplify* $-(4y - 3z)$.

$$-(4y - 3z) = -1(4y - 3z)$$
$$= -4y + 3z.$$

Ex. 3.—*Simplify* $2x(x^2 - 2xy + 3y^2)$.

We must multiply each of the quantities x^2, $-2xy$ and $3y^2$ by $2x$.

$$\therefore 2x(x^2 - 2xy + 3y^2) = 2x^3 - 4x^2y + 6xy^2.$$

EXAMPLES V (a).

1. $2(x + 1)$.
2. $5(b - 2)$.
3. $-4(v + 2)$.
4. $-6(m - 5)$.
5. $2(4 + a)$.
6. $3(5 - a^2)$.
7. $-4(6 + c)$.
8. $-7(-t^3 + 2)$.
9. $-8(-st - 3)$.
10. $4(x + y)$.
11. $-5(y + 4z)$.
12. $-6(2a^2 - 3b^2)$.
13. $3(p - 2q - 4r)$.
14. $-9(2f - 3g - h)$.
15. $a(y + z)$.
16. $b(m + 2n)$.
17. $d(3r - s)$.
18. $-m(x + 5y)$.
19. $-p(x + 3y)$.
20. $k(-s - 2t)$.
21. $-x(-2a - 4b)$.
22. $3t(2a - 5b)$.
23. $-4s(3u - 5v + 2w)$.
24. $x(x + y)$.
25. $y(x - y)$.
26. $a^2(a + b)$.
27. $-b^3(a^2 - b^2)$.
28. $-2w(w^2 - wx)$.
29. $pq(p^2 - q^2)$.
30. $-3ms(m^3 - 2s^3)$.
31. $yz(2y^2 - 3yz - 4z^2)$.
32. $3ab(a^2 - 4ab + 5b^2)$.
33. $-2bc(3b^2 + 2bc - 7c^2)$.

III. The Simplification of More Complicated Expressions.

Ex. 1.—*Simplify* $5(2x - 3y) - 3(x - 4y)$.

Here we have to multiply $2x - 3y$ by 5 and $x - 4y$ by 3 and subtract the second result from the first.

Since $5(2x - 3y) = 10x - 15y$
and $3(x - 4y) = 3x - 12y$

$\therefore$ by subtraction $5(2x - 3y) - 3(x - 4y)$

$$= 10x - 15y - 3x + 12y \quad . \quad . \quad . \quad (a)$$
$$= 7x - 3y.$$

Examination of line (a) shows that its terms are obtained by multiplying the quantities $2x$ and $-3y$ by 5, and the

quantities x and $-4y$ by -3. This process is always applied to obtain the terms of an expression as it appears after removal of brackets.

Ex. 2.—*Simplify* $2x(3x - 5y) + 4y(x - 2y)$.

On multiplying $3x$ and $-5y$ by $2x$, and x and $-2y$ by $+4y$, we have

$$\begin{aligned} 2x(3x - 5y) + 4y(x - 2y) &= 6x^2 - 10xy + 4xy - 8y^2 \\ &= 6x^2 - 6xy - 8y^2. \end{aligned}$$

Ex. 3.—*Simplify*

$$\{4(4p - 3q) - 2(7p - 5q)\} - \{6(2p + q) + (-3p - 4q)\}$$

$$\{4(4p - 3q) - 2(7p - 5q)\} - \{6(2p + q) + (-3p - 4q)\}$$

$$= \{16p - 12q - 14p + 10q\} - \{12p + 6q - 3p - 4q\}$$

[*N.B.* $(-3p - 4q) = 1(-3p - 4q)$]

$$= \{2p - 2q\} - \{9p + 2q\}$$

$$= 2p - 2q - 9p - 2q$$

$$= \mathbf{-7p - 4q}.$$

EXAMPLES V (b).

Simplify :—

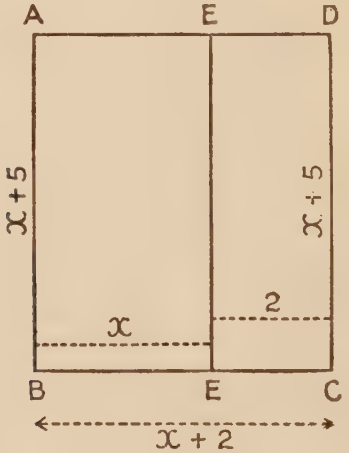
1. $5(x + 2) + 2(x - 3)$
2. $7(y - 4) + 2(3y - 1)$.
3. $6(2pz - 3) - 3(3pz + 4)$.
4. $8(4t^2 - 2) - 4(2t^2 - 5)$.
5. $4(5mn - 6) - 10(2mn - 3)$.
6. $9(4 - 2x) - 3(12 - 7x - 3x)$.
7. $7(2a - 4b) + 4(a + b)$.
8. $6(5b - 2c) - 4(7b + c)$.
9. $5(8y + 3z) - 7(5y)$.
10. $9(4p - 5q) - 6(-7q)$.
11. $3(5t - 3u) + 2(-6t)$.
12. $2(7a - 3b) + (-4b)$.
13. $4(3b - 7c) - (-9b)$.
14. $3(2m - 3n) + (n - 4m)$.
15. $4(3b^3 - 2c^3) - (9b^3 - 12c^3)$.
16. $-7(4p + 3q) - (-10q - 8p)$.
17. $7(6a - b) - 2(4a - 3b) - 4(5a + 2b)$.
18. $-9(2m^2 + 3n^2) + 6(4m^2 + 5n^2) - 2(6m^2 - 7n^2)$.
19. $x(x - 2y) + y(3x - 4y)$.
20. $s^2(s^2 - t^2) - t^2(2s^2 + 4t^2)$.
21. $a(a + b - c) - b(a - c)$.
22. $x(y - z) - y(x - z) + z(x - y)$.
23. $a(a - b - c) - b(-a - b + c) - c(-a - b - c)$.
24. $\{5(a + b) + 4(a - b)\} + \{3(a - 2b) - (2a - 5b)\}$.
25. $(\overline{6x - 3y} - \overline{4x + 2y}) - (\overline{7x + 3y} - \overline{5x - 4y})$.
26. $(7x^2 - \overline{4x^2 + y^2}) + (\overline{9x^2 - 2y^2} - 6y^2)$.
27. $\{7(3m - 4n) - 2(8m - 9n)\} - \{6(2m - n) - 5n\}$.
28. $\{-2(y - 3z) - (6y + 8z)\} - \{-(5y - z) + 5(2y + 4z)\}$.
29. $\{6(4a^2 - 3b^2) + 5(-3a^2 + 2b^2)\} - \{8(3a^2 + 4b^2) - 9(2a^2 + 5b^2)\}$.
30. $\{a(a + b + c) - b(a - b + c)\} + \{b(-a - b - c) - c(a - b - c)\}$.

CHAPTER VI.

LONG MULTIPLICATION AND DIVISION.

I. Long Multiplication.

Let us suppose that we have to multiply $x + 5$ by $x + 2$. In the diagram let AB $= x + 5$, BE $= x$ and EC $= 2$, so that BC $= x + 2$.



Then $(x + 5)(x + 2)$ = rect. AC
$= x(x + 5) + 2(x + 5)$
$= x^2 + 5x + 2x + 10$
$= x^2 + 7x + 10.$

We see therefore that we multiply $x + 5$ by x, and by 2, and add the results. The work of the multiplication may be set down as follows:—

$$\begin{array}{l} x+5 \\ x+2 \\ \hline x^2+5x \quad \text{(Result of multiplying } x+5 \text{ by } x\text{)} \\ \quad 2x+10 \quad (\quad ,, \quad ,, \quad x+5 \text{ by } 2\text{)} \\ \hline x^2+7x+10 \quad \text{(Addition of above results).} \end{array}$$

Ex.—*Multiply* $2x^2 - 3xy - 4y^2$ by $x^2 + 2xy - 5y^2$.

Extending the principles explained above, we multiply $2x^2 - 3xy - 4y^2$ by x^2, $2xy$, and $-5y^2$ and then add the results.

$$\begin{array}{l} 2x^2 - 3xy - 4y^2 \\ \quad x^2 + 2xy - 5y^2 \\ \hline 2x^4 - 3x^3y - 4x^2y^2 \\ \quad\quad 4x^3y - 6x^2y^2 - 8xy^3 \\ \quad\quad\quad - 10x^2y^2 + 15xy^3 + 20y^4 \\ \hline 2x^4 + x^3y - 20x^2y^2 + 7xy^3 + 20y^4. \end{array}$$

EXAMPLES VI (a).

Multiply :—

1. $x + 3$ by $x + 2$.
2. $x - 3$ by $x + 4$.
3. $x - 9$ by $x - 8$.
4. $y + 6$ by $y - 7$.
5. $10 - p$ by $3 + p$.
6. $2x + 1$ by $x + 1$.
7. $3x - 2$ by $x - 4$.
8. $2x - 3$ by $3x + 4$.
9. $4m + 6$ by $2m - 3$.
10. $7t + 4$ by $4t - 1$.
11. $3a + b$ by $a + 3b$.
12. $a + 4b$ by $a - 4b$.
13. $3m - n$ by $3m - n$.
14. $4 - 5a$ by $2 + a$.
15. $2 + 6a$ by $3 - 5a$.
16. $2y - 8z$ by $5y - 7z$.
17. $x^2 - y^2$ by $x^2 - 5y^2$.
18. $a^2 - 2b^2$ by $2a^2 - 3b^2$.
19. $x^2 + x + 4$ by $x + 1$.
20. $x^2 - 7x + 3$ by $x + 2$.
21. $x^2 - 9x - 3$ by $x + 3$.
22. $x^2 + 8x - 6$ by $x - 2$.
23. $a^2 - 6a - 5$ by $a - 4$.
24. $y^2 - 7y - 3$ by $-y + 5$.
25. $x^2 + 2x - 6$ by $3x + 4$.
26. $b^2 - 3b + 5$ by $2b - 3$.
27. $c^2 - 5c - 2$ by $2 - 4c$.
28. $2w^2 - 7w - 3$ by $2w + 3$.
29. $2x^2 + 4x - 6$ by $3x - 4$.
30. $5m^2 - 5m + 3$ by $2m - 3$.
31. $3r^2 + 8r - 6$ by $5 - 3r$.
32. $3y^2 - 5y - 1$ by $4y - 3$.
33. $1 + 2m - m^2$ by $1 - 4m$.
34. $3 - 4a + 5a^2$ by $4 + 2a$.
35. $x^2 - xy + y^2$ by $x + y$.
36. $m^2 + 3mn + 9n^2$ by $m - 3n$.

37. $2y^2 + 4yz - 6z^2$ by $3y - 4z$.
38. $3a^2 - ab - 5b^2$ by $3b - 2a$.
39. $-4b^2 + 8bc - c^2$ by $-3b - 2c$.
40. $2p^3 - 4p^2 - 2p + 3$ by $2p - 5$.
41. $3x^3 - x^2 + 7x - 5$ by $2x + 4$.
42. $4x^3 + 5x^2 - x - 6$ by $4x - 2$.
43. $2 - 3a + 4a^2 - a^3$ by $4 - 7a$.
44. $x^2 + x + 1$ by $x^2 + 2x + 3$.
45. $2a^2 - a - 3$ by $a^2 - a + 4$.
46. $b^2 + 2b + 2$ by $b^2 - 2b + 2$.
47. $2y^2 - y - 1$ by $2y^2 + 3y - 5$.
48. $3c^2 + 2c - 4$ by $-c^2 + 4c - 2$.
49. $4z^2 - 6z - 8$ by $2z^2 + 3z - 1$.
50. $x^2 - 2x + 5$ by $3x^2 - x - 4$.
51. $x^2 + 2xy + 3y^2$ by $x^2 - 2xy + 3y^2$.
52. $y^2 + 4yz - 2z^2$ by $2y^2 - yz + z^2$.
53. $3a^2 - 5ab + 4b^2$ by $2a^2 - ab - 2b^2$.
54. $3b^2 - 4bc - 2c^2$ by $4b^2 - 2bc - 5c^2$.
55. $4 - a - a^2$ by $3 + 2a - a^2$.
56. $x^3 + 2x + 1$ by $x^2 - x - 4$.
57. $2x - 3y + z$ by $2x + 3y - z$.
58. $a^2 + b^2 + c^2 - ab + bc + ca$ by $a + b - c$.

59. Multiply $5x^3 + 2x - 7$ by $x^3 - 4x + 3$. Check the result by taking $x = 2$.

60. Multiply $4a^2 - 3ab + 6b^2$ by $2a^2 + ab - b^2$. Verify your result by putting $a = 2$ and $b = 1$.

II. Long Division.

Ex. 1.—*Divide* $x^2 + 8x + 12$ by $x + 6$.

Put otherwise this means "By what must we multiply $x + 6$ to obtain $x^2 + 8x + 12$"?

(i) If we multiply $x + 6$ by x we get $x^2 + 6x$.
(ii) Deducting $x^2 + 6x$ from $x^2 + 8x + 12$ leaves $2x + 12$.
(iii) Multiplication of $x + 6$ by 2 gives $2x + 12$.

Hence we must multiply $x + 6$ by $x + 2$.

The work of division is usually set down as follows, following the plan adopted in arithmetic.

$$
\begin{array}{rl}
 & x + 2 \\
\hline
x + 6 \,) & x^2 + 8x + 12 \\
 & x^2 + 6x \qquad\qquad \text{Step (i) as explained above} \\
 & \hline
 & 2x + 12 \qquad \text{,, (ii) ,, ,,} \\
 & 2x + 12 \qquad \text{,, (iii) ,, ,,} \\
 & \hline
\end{array}
$$

N.B. $x^2 \div x = x$ the first term of the quotient.
$2x \div x = 2$,, last ,, ,, ,,

Ex. 2.—*Divide* $m^3 - 3m^2n + 3mn^2 - n^3$ by $m - n$.

$$
\begin{array}{r|l}
m - n & m^2 - 2mn + n^2 \\
\hline
m^3 - 3m^2n + 3mn^2 - n^3 & \text{NOTES: } m^3 \div m = m^2. \\
m^3 - m^2n & \\
\hline
-2m^2n + 3mn^2 & -2m^2n \div m = -2mn. \\
-2m^2n + 2mn^2 & \\
\hline
mn^2 - n^3 & mn^2 \div m = n^2. \\
mn^2 - n^3 & \\
\hline
\end{array}
$$

Ex. 3.—*Divide* $-9 + 5x^2 + 2x^4 + 7x + 5x^3$ by $3x + 4 + 2x^2$.

We first arrange the expressions in ascending or descending powers of x, and then perform the division.

$$
\begin{array}{r|l}
2x^2 + 3x + 4 & x^2 + x - 1 \\
\hline
2x^4 + 5x^3 + 5x^2 + 7x - 9 & \\
2x^4 + 3x^3 + 4x^2 & \\
\hline
2x^3 + x^2 + 7x & \\
2x^3 + 3x^2 + 4x & \\
\hline
-2x^2 + 3x - 9 & \\
-2x^2 - 3x - 4 & \\
\hline
6x - 5 & \text{(Remainder.)} \\
\hline
\end{array}
$$

EXAMPLES VI (b).

Divide :—

1. $x^2 + 7x + 12$ by $x + 3$.
2. $y^2 - y - 12$ by $y - 4$.
3. $a^2 - 12a + 27$ by $a - 3$.
4. $b^2 + 4b - 32$ by $b - 4$.
5. $p^2 + 4p - 77$ by $p + 11$.
6. $2x^2 + 7x + 5$ by $x + 1$.
7. $3x^2 + 4x - 4$ by $x + 2$.
8. $4y^2 - 18y + 20$ by $2y - 5$.
9. $6m^2 - 5m - 6$ by $2m - 3$.
10. $6c^2 - 7c - 3$ by $3c + 1$.
11. $4b^2 - 4b - 15$ by $2b - 5$.
12. $9m^2 - 49$ by $3m + 7$.
13. $5p^2 - 28p + 15$ by $p - 5$.
14. $x^2 - 6x + 2$ by $x + 2$.
15. $a^2 + 7a + 3$ by $a - 1$.
16. $x^2 - 9x - 5$ by $x + 2$.
17. $p^2 - 2p - 30$ by $p - 7$.
18. $9c^2 - 6c - 8$ by $3c - 4$.
19. $20 - 3p - 2p^2$ by $4 + p$.
20. $18 - 9w - 2w^2$ by $3 - 2w$.

21. $6 + 2x^2 - 34$ by $x^2 + 7$.
22. $m^2 + mn - 5n^2$ by $6m - 5n$.
23. $6a^2 + 5ab - 6b^2$ by $3a - 2b$.
24. $12s^2 + 11t - 56t^2$ by $4s - 7t$.
25. $5a^2 - 21ab + 5b^2$ by $5a - b$.
26. $x^3 + 5x^2 + 7x + 3$ by $x + 3$.
27. $x^3 - x^2 + x - 6$ by $x - 2$.
28. $b^3 - b^2 - 10b - 8$ by $b + 2$.
29. $3 - 2x^2 + x^3 - 8x$ by $-1 + x$.
30. $t^3 + 8$ by $t + 2$.
31. $y^3 - 27$ by $y - 3$.
32. $6z^3 + 11z^2 + 15z + 4$ by $3z + 1$.
33. $12r^3 + 8 - 18r - 2r^2$ by $-2 + 4r$.
34. $14a^2 - 12 + 30a^3 - 23a$ by $4 + 5a$.
35. $x^4 - y^4$ by $x - y$.
36. $x^3 - 4x^2 - 2x - 15$ by $x^2 + x + 3$.
37. $6x^3 - 19x^2 + 8x + 5$ by $3x^2 - 2x - 1$.
38. $16m^3 + 8m^2 - 3m - 1$ by $4m^2 + m - 1$.
39. $3m^3 - 19m^2 - 41m - 18$ by $m^2 - 7m - 9$.
40. $9m^3 - 3m^2 - 24m + 10$ by $3m^2 - 6m + 2$.
41. $5m^3 + 3m^2 - 4m - 14$ by $m^2 + 2m + 2$.
42. $8b^3 - 28b^2c - 38bc^2 + 9c^3$ by $4b^2 + 4bc - c^2$.
43. $x^3 + 4x^2 + 6x + 8$ by $x^2 + x + 3$.
44. $a^3 + 5a^2 - 7a - 3$ by $a^2 - 2a + 6$.
45. $y^3 - 9y^2 - 2y + 3$ by $y^2 - 5y - 8$.
46. $10z^3 - 26z^2 + 24$ by $5z^2 - 3z - 6$.
47. $7x + 5x^3 + 2x^4 + 4 + 9x^2$ by $3x + 4 + 2x^2$.
48. $x^2 + 1 + x^4$ by $1 + x^2 - x$.
49. $y^4 - 10y^2 + 9$ by $4y + y^2 + 3$.
50. $-2 - 17a^2 + 5a^3 + 11a + 3a^4$ by $-4a + 3a^2 + 1$.
51. $40 - 4c - 4c^2 + 4c^3$ by $5 - 3c + c^2$.
52. Multiply $2x^2 + 7x + 6$ by $2x^2 - 3x + 1$ and divide the result by $2x^2 + 3x - 2$.
53. Find the quotient and remainder on dividing

$$8x^3 - 18x^2 + 15x + 3 \text{ by } 4x^2 - 7x + 3.$$

Check your results by putting $x = 2$.

54. Find the quotient and remainder on dividing $x^4 - 50x + 10$ by $x^2 - 4x + 1$. Verify your result by putting $x = 3$.
55. If $x^4 + 4x^2 - x + a$ is divisible by $x^2 - x + 2$, what must be the value of a?
56. On dividing a certain expression by $2x^2 - x - 3$ the quotient is $x + 5$ and the remainder $-x + 13$. Find the expression and show that it is divisible by $x - 1$.
57. Show that if $x^6 - y^6$ be divided by $x^2 - y^2$ the quotient is equal to $(x^2 + xy + y^2)(x^2 - xy + y^2)$.

CHAPTER VII.

EQUATIONS OF THE FIRST DEGREE IN ONE UNKNOWN.

I. EQUATIONS arise in connection with various forms of mathematical work. Below we give several examples of equations.

(1) $4x = 28$ (2) $6x - 7 = 2y + 5$ (3) $t^2 = 8t + 20$
(4) $5x - 3y + 2 = x + 4y - 3$ (5) $l^2 - 4lm = 9 - m^2$.

We see, then, that an equation states an equality between two independent expressions, and may contain one or more letters of unknown value. The statements are not true for all numbers. Take, for example, equation (3), and let $t = 4$. Then $t^2 = 16$ and $8t + 20 = 32 + 20 = 52$, so that when $t = 4$, t^2 is not equal to $8t + 20$. But it is true when $t = 10$ for then $t^2 = 100$ and $8t + 20 = 80 + 20 = 100$. It is also true when $t = -2$ since, in that case $t^2 = 4$ and $8t + 20 = -16 + 20 = 4$. Again, equation (1) is true when $x = 7$ (since $4 \times 7 = 28$) but is true for no other number, so that we see that some equations are satisfied by one number while others may be satisfied by several numbers.

A statement such as the following is called an identity, namely $(x + 2)(x + 7) = x^2 + 9x + 14$. It expresses the fact that two forms of the same expression are equal, and is therefore obviously true no matter what value we give to x.

In this chapter we shall deal with a certain class of equation containing one unknown and true for only one value of the same. These equations are called equations of the first degree in one unknown. When we have found the number for which any such equation is true **we are said to have solved the equation,** and the number is called the **solution or root.**

II. We proceed to show how to solve some typical equations of the first degree.

When solving these equations we shall use the following principles, namely, that if two quantities be equal then

(i) If we multiply them by the same number the results are equal.

(ii) If we divide them by the same number the results are equal.

(iii) If we add the same number to each of them the results are equal.

(iv) If we subtract the same number from each of them the results are equal.

Ex. 1.—*Solve* $5x = 20$.
On dividing each side by 5 we get $x = 4$.

Ex. 2.—*Solve* $3x + 20 = 2$.
From each side of the equation subtract 20

$$\therefore 3x + 20 - 20 = 2 - 20$$
$$\therefore 3x = -18.$$

Divide both sides by 3

$$\therefore x = -6.$$

Ex. 3.—*Solve* $7x - 3 = 5x + 9$.
(i) To remove x terms from the Right-Hand Side (R.H.S.) subtract $5x$ from each side of the equation

$$\therefore 7x - 3 - 5x = 9 \quad . \quad . \quad . \quad (a)$$

(ii) To remove numerical terms from the Left-Hand Side (L.H.S.) add 3 to each side of the equation (a)

$$\therefore 7x - 5x = 9 + 3 \quad . \quad . \quad . \quad (b)$$
$$\therefore 2x = 12$$
$$\therefore x = 6.$$

If we compare equation (b) with the original equation we see that the solution may be effected by collecting all the x-terms to one side and all the numerical terms to the other, accompanying a change of side by a change of sign.

When solving equations pupils should not employ the mechanical process just described, for a considerable time.

Ex. 4.—*Solve* $13(w - 2) - 2(w - 4) = 8(w - 1) - w + 6$
On removing brackets we have

$$13w - 26 - 2w + 8 = 8w - 8 - w + 6 \quad . \quad (a)$$
$$\therefore 11w - 18 = 7w - 2$$

Transpose terms $\therefore 11w - 7w = -2 + 18$ . . . (*b*)
$\therefore 4w = 16.$
$\therefore w = 4.$

Remarks (i) After a little practice step (*b*) may be omitted.
(ii) If preferred transposition of terms may be effected without first simplifying equation (*a*).

EXAMPLES VII (a).

Solve :—

1. $3y = 12.$
2. $6t = 18.$
3. $2v = 7.$
4. $5m = 28.$
5. $4n = 0.$
6. $5t = -30.$
7. $3x = -21.$
8. $8w = -32.$
9. $6v = -20.$
10. $9u = -21.$
11. $-2a = 6.$
12. $-4b = 32.$
13. $-3c = 7.$
14. $-d = -24.$
15. $-5z = -10.$
16. $-7y = -28.$
17. $-9x = -36.$
18. $-w = -56.$
19. $3x = a$ (for x).
20. $2t = 4b$ (for t).
21. $-w = 30a$ (for w).
22. $vt = 9$ (for t).
23. $xy = 12$ (for x).
24. $ft = v$ (for t).
25. $x + 3 = 5.$
26. $z + 2 = 9.$
27. $a - 2 = 3.$
28. $y - 4 = 5.$
29. $a + 6 = 14.$
30. $m - 9 = -2.$
31. $w - 8 = -3.$
32. $x - 3 = -7.$
33. $z - 4 = -10.$
34. $p + 3 = -9.$
35. $-t - 4 = 7.$
36. $-n + 7 = -11.$
37. $2x + 7 = 13.$
38. $2z - 10 = 8.$
39. $6t + 21 = 3.$
40. $5p - 4 = -24.$
41. $2x + 11 = 0.$
42. $10c - 6 = -21.$
43. $23 = 3 - 4x.$
44. $-19 = 6 + 5a.$
45. $8 - 3v = 8.$
46. $2(x - 5) = 7.$
47. $3(2z + 3) = 27.$
48. $2(5x + 11) = 7.$
49. $5(3 - x) = 15.$
50. $7x - 15 = 4x.$
51. $9t = 7t - 8.$
52. $6w = 13 - 7w.$
53. $3y + 14 = -4y.$
54. $-11s = -7s + 20.$
55. $2x - 4 = a$ (for x).
56. $3y + 6 = b$ (for y).
57. $7w - c = 8$ (for w).
58. $ax - 2 = b$ (for x).
59. $az - b = 4$ (for z).
60. $ct - d = 9$ (for t).

EXAMPLES VII (b).

1. $5x - 2 = 3x + 6.$
2. $12y - 4 = 7y + 11.$
3. $7t + 3 = 3t - 9.$
4. $10w - 2 = 3w - 16.$
5. $9 - z = 11z - 15.$
6. $40 - 3y = 4 - 9y.$

7. $8 + 5p = 32 + 11p$.
8. $6 - 8s = 11 - 10s$.
9. $13 - 2t = t - 5$.
10. $9v - 7 = 7v + 8$.
11. $16 - 8t = 4 - 3t$.
12. $5(x - 2) = 2(x - 8)$.
13. $8(3 - z) = 6(z + 4)$.
14. $9(8 - r) = 5(12 - r)$.
15. $11(2w + 3) = 10(3w + 5)$.
16. $3(3 - 4s) = 2(3 - 5s)$.
17. $7(3 + 5x) = 3(7 - 9x)$.
18. $6(3m + 2) = 4(4m - 5)$.
19. $12x + 5 - 4x + 3 = 11x + 26 - 5x - 6$.
20. $15y + 4 - 6y - 9 = 12y - 18 - 7y - 11$.
21. $7t - 16 + 3t + 4 = 8t + 21 + 5t - 12$.
22. $19c - 5 - 10c + 8 = 13c + 17 - 11c - 6$.
23. $6 - 7b - 28 - 9b = 13 - 17b - 5 + 7b$.
24. $6 - 19p + 17 + 18p = 10 - 5p - 3 + 7p$.
25. $9(v - 1) - 2(v - 2) = 6(3v - 2) - 6(v + 1) - 1$.
26. $3(y - 5) - 2(y - 5) = 4(3y - 1) - 2(8y - 7)$.
27. $4(2r + 3) - 3(3 - r) = 2(r - 6) - 2(7 - 5r)$.
28. $3(5x - 4) - 5(x - 1) - 2(5x + 9) = 2(x + 3) + 3(3 - 2x)$.
29. $7(4m - 5) - 5(3 - 2m) - 3(9m - 8) = 9(3m - 4) + 10$.
30. $4(5b - 3) - 6(2b + 7) = 3(4 - 7b) - 5(8 - 6b) + 3b$.

VI. Equations Containing Fractions.

Ex. 1.—*Solve* $\frac{3}{4}x = -6$.

Multiply both sides by 4.

$\therefore 3x = -24$.

$\therefore x = -8$.

Ex. 2.—*Solve* $\frac{3}{2}p - \frac{2}{3} = \frac{5}{6}$.

Multiply both sides by 6, the L.C.M. of the denominators.

$$\therefore \left(6 \times \frac{3}{2}p\right) - \left(6 \times \frac{2}{3}\right) = 6 \times \frac{5}{6}.$$

$\therefore 9p - 4 = 5$.

Whence $p = 1$.

Ex. 3.—*Solve* $\frac{5r + 1}{6} = \frac{7r - 1}{9}$.

Multiply both sides by 18, the least common denominator.

$$\therefore 18 \times \frac{1}{6}(5r + 1) = 18 \times \frac{1}{9}(7r - 1).$$

$$\left[N.B.\ \frac{5r + 1}{6} = \frac{1}{6}(5r + 1).\right]$$

$\therefore 3(5r + 1) = 2(7r - 1)$.

Whence $r = -5$.

Ex. 4.—*Solve* $\frac{4}{5}(7y + 2) - \frac{1}{6}(7y - 4) = \frac{7}{10}(3y + 8) + 6.$

Multiply both sides by 30, the L.C.M. of the denominators.

$$\therefore 24(7y + 2) - 5(7y - 4) = 21(3y + 8) + 180.$$

[*N.B.* $30 \times \frac{4}{5}(7y + 2) = 24(7y + 2)$.]

Whence $y = 4$.

Verification :—

When $y = 4$, L.H.S. (left-hand side) $= \frac{4}{5}(28 + 2) - \frac{1}{6}(28 - 4)$

$$= \frac{4}{5} \times 30 - \frac{1}{6} \times 24$$
$$= 24 - 4 = 20$$

and R.H.S. (right-hand side) $= \frac{7}{10}(12 + 8) + 6$

$$= \frac{7}{10} \times 20 + 6$$
$$= 14 + 6 = 20.$$

$\therefore$ L.H.S. = R.H.S.

N.B.—It is always the original equation that must be used in the verification, and not any of the derived equations.

Ex. 5.—*Solve* $6{\cdot}4c - 1{\cdot}1 - 3{\cdot}6c - 7{\cdot}8 = 7{\cdot}3c + 4{\cdot}6.$

On simplifying the L.H.S. we have

$$2{\cdot}8c - 8{\cdot}9 = 7{\cdot}3c + 4{\cdot}6.$$
$$\therefore -4{\cdot}5c = 13{\cdot}5$$
$$\therefore c = \frac{13{\cdot}5}{-4{\cdot}5} = -3.$$

EXAMPLES VII (c).

Solve :—

1. $\frac{1}{2}x = 3.$
2. $\frac{x}{5} = -6.$
3. $\frac{2}{3}m = 12.$
4. $\frac{4}{3}z = -24.$
5. $-\frac{5}{2}t = 15.$
6. $-\frac{7}{8}y = -14.$
7. $-\frac{11}{10}w = -22.$
8. $\frac{2}{5}x = -3.$

9. $\frac{x}{2} + 6 = 8.$

10. $\frac{3}{4}z - 6 = -3.$

11. $\frac{2}{3}t - 5 = -11.$

12. $\frac{4}{5}w + 10 = -2.$

13. $\frac{x}{2} + \frac{3}{4} = 3.$

14. $\frac{2}{3}v - \frac{5}{6} = 5.$

15. $\frac{3}{2}y - \frac{2}{3} = \frac{5}{6}.$

16. $\frac{5}{8}z + \frac{3}{4} = \frac{1}{6}.$

17. $\frac{2}{5} - \frac{4}{3}t = \frac{9}{10}.$

18. $\frac{x}{2} - 5 = \frac{x}{4} + 2.$

19. $\frac{2}{3}z - 4 = \frac{1}{6}z + 1.$

20. $\frac{5}{6}y + 3 = \frac{4}{9}y - 4.$

21. $\frac{1}{2}y - \frac{1}{3} = \frac{3}{4}y - \frac{5}{6}.$

22. $\frac{x}{3} + \frac{1}{9} = \frac{2}{9}x - \frac{2}{3}.$

23. $\frac{9}{8}t - \frac{2}{3} = \frac{3}{4}t - 1\frac{1}{6}.$

24. $\frac{15}{4}r + 3 = \frac{14}{3}r - \frac{2}{3}.$

25. $\frac{1}{3}w + 6 = \frac{1}{5}(w - 8).$

26. $\frac{1}{5}(2x + 4) = \frac{1}{6}(3x + 7).$

27. $\frac{x - 3}{2} = \frac{x + 4}{4}.$

28. $\frac{z + 5}{3} = \frac{z - 7}{4}.$

29. $\frac{t - 8}{4} = \frac{t - 7}{6}.$

30. $\frac{1}{3}(w - 6) = \frac{1}{5}(w - 8).$

31. $\frac{1}{9}(2x - 3) = \frac{1}{7}(6 + x).$

32. $\frac{1}{4}(4y - 3) = \frac{1}{7}(6y - 5).$

33. $\frac{3}{2}(3v + 1) = \frac{5}{4}(5v - 3).$

34. $\frac{4}{3}(2x + 25) = \frac{5}{6}(5x + 49).$

35. $\frac{5}{2}(3t + 20) = \frac{2}{9}(70 - 5t).$

36. $\frac{9(1 - z)}{4} = \frac{6(1 - 2z)}{5}.$

37. $\frac{5(4 - 2x)}{3} = \frac{3(1 - 7x)}{5}.$

38. $\frac{3}{4}(x - 2) - 2x = 6.$

39. $\frac{2}{3}(t + 7) + 12 = 4t.$

40. $9x + 3 - \frac{5}{6}(7x - 4) = 0.$

41. $\frac{4}{5}(3z + 4) = 2z - \frac{5}{6}.$

42. $\frac{3}{2}(3r - 5) - 9 - \frac{8}{3}r = 0.$

43. $\frac{7y}{6} - \frac{5(3y - 4)}{4} = \frac{46}{3}.$

44. $\frac{t - 2}{4} - \frac{t + 5}{8} = 2.$

45. $\frac{y + 8}{3} - \frac{y - 2}{9} = 4.$

46. $\frac{1}{2}(x - 4) = \frac{1}{7}(x + 3) - 1.$

47. $\frac{2x + 5}{3} - \frac{x + 2}{6} = 1.$

48. $\frac{3}{2}(4x + 1) + \frac{5}{4}(x - 2) = 13\frac{1}{4}.$

49. $\frac{6w+1}{5} - \frac{2w+1}{35} = \frac{5w-6}{7}$.

50. $\frac{3(t+2)}{4} + \frac{5(2t-1)}{6} = \frac{7(t+7)}{8} - \frac{5}{6}$.

51. $\frac{2}{3}(4v+7) - \frac{4}{9}(5v-4) = \frac{5}{6}(v+4)$.

52. $\frac{3}{4}(5s-1) - \frac{10}{3}(s+1) = \frac{7}{8}(s-1)$.

53. $\frac{1}{4}(w-3) + 10 = \frac{1}{7}(20-3w) + 3$.

54. $\frac{3(3x-4)}{2} - \frac{4(x+4)}{5} = \frac{9(2x-6)}{10}$.

55. $\frac{3}{4}(z-6) = \frac{5}{6}(2z-1) - \frac{11}{15}(4z+5)$.

56. $\frac{3y+1}{2} - \frac{4y+3}{3} - \frac{3y-4}{12} = 0$.

57. $\frac{1}{6}(t+2) - \frac{1}{4}(t-7) - \frac{1}{8}(t+25) = 0$.

58. $\frac{1-2w}{5} - \frac{1-16w}{20} = 1\frac{1}{2} + w$.

59. $\frac{2}{3}(4p-7) - \frac{1}{7}(3p-5) = 4 + \frac{1}{9}(5p-11)$.

60. $\frac{4(3x+1)}{5} - \frac{5x-3}{6} = \frac{3(11x+7)}{10} - \frac{2(16x-3)}{15}$.

EXAMPLES VII (d).

1. $\cdot 2x = 6$.
2. $\cdot 4x = 8$.
3. $\cdot 3y = 1\cdot 2$.
4. $\cdot 5p = 1\cdot 5$.
5. $1\cdot 2x = 4\cdot 8$.
6. $\cdot 6x = -1\cdot 8$.
7. $\cdot 8z = -1\cdot 44$.
8. $\cdot 7x = -4\cdot 2$.
9. $-\cdot 5x = 4\cdot 5$.
10. $-\cdot 6y = -4\cdot 2$.
11. $-1\cdot 3t = -\cdot 91$.
12. $\cdot 4x - 3\cdot 5 = \cdot 8x - 1\cdot 3$.
13. $\cdot 6x + \cdot 9 = \cdot 4x + 1\cdot 5$.
14. $\cdot 7t - 1\cdot 9 = \cdot 9t - 5$.
15. $7\cdot 5y + 4\cdot 4 = 4\cdot 3y - 5\cdot 2$.
16. $1\cdot 5t - 6\cdot 5 = 3\cdot 3t + 6\cdot 7$.
17. $5\cdot 4m - 7\cdot 3 - 1\cdot 3m + 2\cdot 7 = 3\cdot 6m - 1\cdot 8$.
18. $6\cdot 4s - 4\cdot 6 - 7\cdot 3s - 7\cdot 8 = 3\cdot 6s + 1\cdot 1$.
19. $5\cdot 6r - 1\cdot 3 = 6\cdot 7r - 8\cdot 9 + 2\cdot 4r - 2\cdot 9$
20. If $P = mf$ find f when $P = -9$ and $m = 1\cdot 5$.
21. If $r^2 = \frac{q}{vw}$ find q when $r = -1\cdot 5$, $v = 2\cdot 5$ and $w = 4$.
22. If $c(v-1) = xrv$ find v when $c = \frac{4}{5}$, $a = -\frac{9}{10}$ and $r = \frac{5}{6}$.
23. If $23.5x - 7 = 3x + c$ is satisfied when $x = 6$, find the value of c.

24. If $v^2 = u^2 + 2fs$ find the value of s when $v = 10$, $u = 6$ and $f = 4$.
25. If $g = s(g+q)$ find g when $s = 1{\cdot}4$ and $q = 2{\cdot}5$.
26. If $\frac{4x - 6}{3} = \frac{7x - 12}{5}$ find the value of $\frac{2}{3}(2x+3) - \frac{3}{4}(x+2)$.
27. Find the value of c so that the equation $\frac{1}{6}(2y - 3) = \frac{4}{9}(y + 2c) - \frac{5}{3}$ may be satisfied when $y = -10$.
28. If $\frac{x + 10}{2} = 3$, $\frac{y - 6}{3} = 4$ and $\frac{z + 9}{4} = 5$, find the value of $7x - 3y + 6z$.
29. If $\frac{v + 5}{4} = 2$, and $\frac{v + w}{7} = 4$, find w.
30. If $\frac{r - 8}{3} = 2$, $\frac{s + 19}{4} = 3$ and $\frac{r + s + t}{8} = \frac{3}{4}$ find t.
31. If $\frac{1}{r} + \frac{1}{n} = \frac{1}{c}$ find r when $n = \frac{2}{3}$, and $c = \frac{4}{5}$.
32. Solve for l the equation $\frac{bl}{2} = A$.
33. Solve for w the equation $c = \frac{w}{2} + 9$.
34. Solve for x the equation $y = mx + 6$.
35. Solve for z the equation $p = q - wz$.

CHAPTER VIII.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

I. Suppose we are told that x and y are two numbers whose sum is 12. Then we can form the equation $x+y=12$. Now if $x = 1$ then $y = 11$ (since $1 + 11 = 12$); and if $x = 4$ then $y = 8$; if $x = 9$ then $y = 3$; if $x = -5$ then $y = 17$, and so on. We see, therefore, that whenever we name a value for x then there is a corresponding value of y such that the two numbers satisfy the equation $x + y = 12$.

If however we are also told that the difference of the numbers is 2, that is to say that $x - y = 2$, then we shall find that not every two numbers which satisfy our first equation will at the same time, satisfy the second. Thus, for example, while $x = 9$ and $y = 3$ satisfy $x + y = 12$ they do not satisfy $x - y = 2$. Both equations, however, are satisfied when $x = 7$ and $y = 5$.

Equations such as $x + y = 12$; $x - y = 2$; $3x - 2y = 5$; $7x + 3y = 6$; $y = 4x - 3$; $y = \frac{3}{4}x$, etc., are called equations of the first degree in x and y. They contain no powers of the unknowns other than the first and no products such as xy. In this chapter we shall show how to find the numbers which satisfy any pair of independent equations of this type. A pair of such equations is called a pair of **simultaneous equations.**

Examples will best show how simultaneous equations are solved.

Ex. 1.—Solve :—

$$9x + 4y = 14 \quad . \quad . \quad . \quad . \quad (1)$$
$$7x - 6y = 20 \quad . \quad . \quad . \quad . \quad (2)$$

Multiply both sides of equation (1) by 7, and of equation (2) by 9

$$\therefore 63x + 28y = 98 \quad . \quad . \quad . \quad (3)$$
$$63x - 54y = 180 \quad . \quad . \quad . \quad (4)$$

Subtract (4) from (3)

$$\therefore 82y = -82, \qquad \therefore y = -1.$$

Since $y = -1$ then from equation (1) we have $9x - 4 = 14$

whence $x = 2$.

Observe that we have so arranged matters that in equations (3) and (4) the same multiple of x occurs. This has been done so that on subtraction we may have an equation containing y only, which we know how to solve. In doing so we are said to **eliminate** x.

We might, if we pleased, eliminate y instead of x. The numerical coefficients of y are 4 and 6, numbers the L.C.M. of which is 12. On multiplying equation (1) by 3 and equation (2) by 2 we have

$$27x + 12y = 42 \quad . \quad . \quad . \quad (5)$$
and $$14x - 12y = 40 \quad . \quad . \quad . \quad (6)$$

Add (5) and (6) $\therefore 41x = 82, \quad \therefore x = 2.$
Since $x = 2$ then from equation (2) we get $14 - 6y = 20$

whence $y = -1$.

Notes.—After finding one of the unknowns it is immaterial which of our equations we use to derive the value of the remaining unknown.

It is usual to set out the final results as follow:—

x	2
y	-1

When using the method of solution just explained, the pupil, after he has fully simplified the two given equations, should arrange them with the unknowns (x and y say) on the left-

hand side and the numerical term on the right-hand side thus :—

$$ax + by = c \quad . \quad . \quad . \quad . \quad (1)$$
$$dx + ey = f \quad . \quad . \quad . \quad . \quad (2)$$

Ex. 2.—Solve :—

$$y = 1 - 2x \quad . \quad . \quad . \quad . \quad (1)$$
$$4x + 3y = 5 \quad . \quad . \quad . \quad . \quad (2)$$

We shall now show another useful mode of solution. In equation (2) substitute for y the value $1 - 2x$ given in (1)

$$\therefore 4x + 3(1 - 2x) = 5$$
whence $x = -1$.

Substitute for x in (1), $\therefore y = 1 + 2 = 3$.

Solutions

x	-1
y	3

Ex. 3.—*Solve* :—

$$\frac{3}{u} - \frac{2}{v} = 2 \quad . \quad . \quad (1) \qquad \frac{4}{u} - \frac{3}{v} = 1 \quad . \quad . \quad (2)$$

Let $\frac{1}{u} = x$ and $\frac{1}{v} = y$. Equations (1) and (2) then become

$$3x - 2y = 2 \quad . \quad (3) \quad \text{and} \quad 4x - 3y = 1 \quad . \quad . \quad (4)$$

Equations (3) and (4) give $x = 4$ and $y = 5$.

$$\therefore \frac{1}{u} = 4 \qquad \therefore 1 \div \frac{1}{u} = 1 \div 4 \qquad \therefore u = \frac{1}{4}$$

$$\text{and } \frac{1}{v} = 5 \qquad \therefore 1 \div \frac{1}{v} = 1 \div 5 \qquad \therefore v = \frac{1}{5}.$$

Solutions

u	$\frac{1}{4}$
v	$\frac{1}{5}$

EXAMPLES VIII (a).

Solve :—

1. $x + y = 7.$
   $x - y = 3.$
2. $p + 2q = 11.$
   $p - 3q = 1.$
3. $4y + z = 15.$
   $7y - z = 7.$
4. $3x + 2y = 7.$
   $3x + y = 5.$
5. $9a - 2b = 20.$
   $7a - 2b = 12.$
6. $7t + 4w = 2.$
   $7t - 4w = 26.$
7. $x + 2y = 4.$
   $2x + y = 5.$
8. $3c + d = 10.$
   $c - 3d = 10.$
9. $2v + w = - 7.$
   $v + 3w = 4.$
10. $3l - m = - 10.$
    $l + 4m = 14.$
11. $7l + m = 6.$
    $l - 6m = 7.$
12. $p - 3q = 0.$
    $4p - 5q = 14.$
13. $4r + 3s = 18.$
    $2r + 5s = 16.$
14. $6x - 5y = 17.$
    $3x + 4y = 2.$
15. $4x + 5y = 16.$
    $9x + 10y = 31.$
16. $6a - 5b = 9.$
    $4a - 3b = 7.$
17. $9v + 2w = 4.$
    $6v - 3w = 33.$
18. $4p = 3q.$
    $6p - 5q = - 4.$
19. $7y + 2z = 0.$
    $8y + 3z = - 5.$
20. $10s = 3t + 8.$
    $12s = 5t + 4.$
21. $12b = 5c - 11.$
    $3c - 8b = 5.$
22. $y + 2z + 1 = 0.$
    $3y + 1 = 2z.$
23. $3x - 5y + 1 = 0.$
    $4x - 3y = 6.$
24. $2x = 3y - 10.$
    $5x = 2y - 3.$
25. $3p + 4q = 6.$
    $4p = 5q - 23.$
26. $6x = 7y - 4.$
    $4x + 3y = 5.$
27. $3y + 9 = 8w.$
    $2w + y = 11.$
28. $7m - 2p = 22.$
    $3m + 8p + 26 = 0.$
29. $7x = 3y + 46.$
    $3x + 4y + 12 = 0.$
30. $6x = 10y + 9.$
    $9x + 8y = 2.$
31. $11s - r = 10.$
    $11r - 101s = 110.$
32. $9x = 10 - 2y.$
    $7y + 6 = - 11x.$
33. $2p = 9v + 17.$
    $8v = 5p + 1.$
34. $4u = - 7w - 7.$
    $27 - 2w = - 3u.$
35. $7l = 3m - 5.$
    $6l + 5m - 26 = 0.$
36. $7y - 9z + 33 = 0.$
    $9y - 7z + 15 = 0.$
37. $11x + 8y - 28 = 0.$
    $9x - 46 = 5y.$
38. $\frac{5}{m} + \frac{2}{n} = 13.$
    $\frac{4}{m} + \frac{7}{n} = 5.$
39. $\frac{7}{x} + \frac{8}{y} = 11.$
    $\frac{3}{x} + \frac{5}{y} = 0.$
40. $\frac{8}{a} - \frac{7}{b} = 11.$
    $\frac{6}{a} - \frac{5}{b} = 9.$

In the next six examples use the method of direct substitution as explained in Example 2, page 47.

41. $3x + 2y = 5.$
$y = 4x - 3.$

42. $5x - 4y = 2.$
$y = 3x - 4.$

43. $16t + 3v = 14.$
$v = 10 - 4t.$

44. $l = 5m - 7.$
$9l = 22m + 6.$

45. $5p = 4q - 2.$
$p = \frac{3}{4}q.$

46. $4r + 7s + 4 = 0.$
$r = -\frac{2}{3}s.$

47. If $x + y = 7$ and $x - y = 5$, find the value of $8xy$.

48. If $2y + z = 5$ and $3y - z = 5$, find the value of $6y - 5z$.

49. Show that the values of p and q which satisfy both the equations $4p + 3q = 6$ and $6p + 5q = 8$ also satisfy the equation $3p - 4q = 17$.

50. Prove that if $3x + 2y = 10$ and $5x + 4y = 14$, then $x^2 - y^2 = 20$.

51. If $3r - 8s = 13$ and $7r - 6s = 5$, find the value of $8r^2 - rs$.

52. Find the value of $\frac{5t - 4w}{3t - 2w}$ given that $8t - 5w = 9$ and $3t - 4w = 14$.

53. Find the values of y and z which satisfy both the equations $x + 3y - z = 10$ and $2x + y + 2z = 20$ when $x = 3$.

II. Harder Examples.

Ex. 1.—*Solve* $\frac{1}{3}(3x - 2y) + \frac{3}{4}(5x - 4) = 14$ . . . (1)

$$\frac{1}{2}(5y - 3) - \frac{1}{6}(7x - 4y + 2) = 3 \quad . \quad . \quad (2)$$

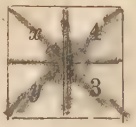
Multiply both sides of equation (1) by 12 and of equation (2) by 6

$$\therefore 4(3x - 2y) + 9(5x - 4) = 168$$
$$\text{and } 3(5y - 3) - (7x - 4y + 2) = 18.$$

Simplification of these results gives

$$57x - 8y = 204 \text{ and } -7x + 19y = 29$$

whence $x = 4$ and $y = 3$ . .



Verification.—When $x = 4$ and $y = 3$ then

$$\frac{1}{3}(3x - 2y) + \frac{3}{4}(5x - 4) = \frac{1}{3}(12 - 6) + \frac{3}{4}(20 - 4).$$
$$= 2 + 12 = 14$$

and $$\frac{1}{2}(5y - 3) - \frac{1}{6}(7x - 4y + 2) = \frac{1}{2}(15 - 3) - \frac{1}{6}(28 - 12 + 2)$$
$$= 6 - 3 = 3.$$

NOTE.—It is the original equations we must use in the verification and not the derived ones.

Ex. 2.—*If the equation* $y = mx + b$ *is satisfied* (i) *when* $x = 3$ *and* $y = 4$ *and* (ii) *when* $x = -2$ *and* $y = 5$, *find* b *and* m. *Hence find* y *when* $x = -7$.

(i) Putting $x = 3$ and $y = 4$ in the equation $y = mx + b$ we have $4 = 3m + b$.

Putting $x = -2$ and $y = 5$ in the equation $y = mx + b$ we have $5 = -2m + b$.

On solving these equations for m and b we find that $m = -\frac{1}{5}$ and $b = 4\frac{3}{5}$.

(ii) Since $y = mx + b$ and since $m = -\frac{1}{5}$ and $b = 4\frac{3}{5}$,

$$\therefore y = -\frac{1}{5}x + \frac{23}{5}.$$

Let $x = -7$ $\quad \therefore y = \frac{7}{5} + \frac{23}{5} = 6$.

EXAMPLES VIII (b).

Solve :—

1. $\frac{x}{3} - \frac{y}{2} = 1.$
   $\frac{x}{6} - \frac{y}{8} = 1.$

2. $\frac{a}{4} + \frac{b}{5} = 2.$
   $\frac{a}{6} - \frac{b}{10} = \frac{1}{6}.$

3. $\frac{2}{3}p - \frac{3}{4}q = 5\frac{1}{2}.$
   $\frac{5}{9}p + \frac{1}{6}q = 3.$

4. $\frac{1}{2}x - \frac{3}{4}y = 9.$
   $\frac{2}{3}x + \frac{5}{2}y = -16.$

5. $\frac{3}{2}r - \frac{1}{4}s = 3\frac{1}{4}.$
   $\frac{1}{4}r + \frac{5}{6}s = -\frac{1}{3}.$

6. $\frac{4}{3}x - \frac{3}{5}y = \frac{7}{15}.$
   $\frac{2}{9}x + \frac{6}{7}y = \frac{25}{63}.$

7. $\frac{1}{2}(2x - y) + \frac{3}{4}x = 6.$

$\frac{3}{4}x - \frac{1}{2}y = 1.$

8. $\frac{1}{3}(w-p) - \frac{1}{6}(2p-3w) = 8.$

$\frac{w}{6} - \frac{p}{3} = 1.$

9. $\frac{x-3y}{3} = \frac{7-6y}{4}.$

$2x + \frac{4}{3}y = \frac{37-12x}{12}.$

10. $\frac{4c+7}{3} = \frac{d+3}{2} = \frac{c+d+3}{3}.$

11. $\frac{x+y}{3} + \frac{2x-y}{4} = 4.$

$\frac{5}{6}(4x + y) - \frac{7}{2}y = 22.$

12. $2y - \frac{z+2}{5} = \frac{5y-4}{2}.$

$\frac{7-z}{2} = z - \frac{5-y}{3}.$

13. $\frac{5}{6}(20-q) - \frac{4}{5}(p+4) = 12.$

$\frac{p+q}{2} + \frac{p-q}{5} = 3.$

14. $\frac{2}{5}\left(3x - 7y + \frac{3}{4}\right) + \frac{1}{2}(2y+9) = 0.$

$\frac{4}{3}(x+y) - \frac{1}{6}(4x+15y) = 1 - 2y.$

15. $\frac{3}{2}(4y+z-6) - \frac{1}{4}(10y-z-2) = 2.$

$\frac{1}{6}(7y+3z+3) - \frac{1}{8}(8y-5z-6) = \frac{1}{2}.$

16. $\frac{5}{6}(4s-t+1) - \frac{4}{9}(7s-2t-2) = 2.$

$\frac{9}{10}(7s - 2t) - \frac{4}{5}(9s - 2t) = -\frac{1}{2}.$

17. $\frac{3v-w}{2} + 3 = \frac{2v+11w}{5}.$

$8 - \frac{5v-2w}{4} = \frac{w}{3} - \frac{v}{2}.$

18. $\frac{1}{6}(2p-q) = \frac{1}{2}(p-5) + \frac{1}{3}q.$

$2p - \frac{1}{9}(10p-q) = \frac{1}{12}(p-q+5).$

19. $\frac{2}{5}(2r-s) = 2 - \frac{3r}{10}.$

$\frac{1}{2}(5r-s+1) = 6s - \frac{3}{4}(7s+1).$

20. $\frac{3x-5y}{8} - \frac{x+11y}{5} = 9.$

$\frac{2x+10y}{15} - \frac{x+25y}{20} = 2.$

21. $\cdot 4x + 2\cdot 1y = 4.6.$
$\cdot 2x + \cdot 5y = 1\cdot 2.$

22. $1\cdot 4b - \cdot 3c = 7\cdot 9.$
$\cdot 7b + \cdot 9c = \cdot 8.$

23. $2\cdot 4y - 1\cdot 6z = 10\cdot 4.$
$\cdot 6y + \cdot 5z = \cdot 8.$

24. $1\cdot 7r - 1\cdot 6s = 10\cdot 9$
$9r - 1\cdot 2s = 6\cdot 3.$

25. $3\cdot 5t - 6\cdot 6w = 25\cdot 6.$
$2\cdot 5t - 4\cdot 2w = 15\cdot 2.$

26. $\cdot 9a + 4\cdot 4b = 5\cdot 5.$
$\cdot 6a + 1\cdot 6b = 4\cdot 5.$

27. Solve $(x+2)(y+3)=(x+1)(y+5)$, $5x-3y=1$.
28. If $r+s=10$ and $r-s=6$ find t if $2r-3s+6t=4$.
29. If $pqr=-28$ find r given that $3p+q=8$ and $3p-q=2$.
30. If $5u-v=4$; $v=3u$; and $\frac{v}{w}=2$ find u, v, and w.
31. If $3m+4n=2$; $3m-n=7$; and $\frac{lm}{n}=8$ find l, m, and n.
32. The equation $lx+my=8$ is satisfied when (i) $x=2$ and $y=1$ and (ii) $x=4$ and $y=6$. Find the values of l and m.
33. If $7x=3y-3$ and $8x-6=5y$ prove that
$$A(x-2y-9)+B(6x-y+12)=0.$$
34. If $3u+v=4$; $u-2v=6$ and $au+bv=10$ prove that $a-b=5$.
35. The equation $x^2+px+q=0$ is satisfied when $x=2$ and $x=4$. Find the values of p and q.
36. The equation $lx+my=2$ is satisfied (i) when $x=4$ and $y=-2$; (ii) when $x=-3$ and $y=1$. Find the values of l and m.
37. If $4x+3y=15c$ and $5x+2y=3c$, find x and y.
38. If $4x-y=2c-6$ and $5x+y=7c-3$, find x and y.
39. The equation $y=\frac{ax+b}{x-2}$ is satisfied (i) when $x=3$ and $y=5$; (ii) $x=4$ and $y=4$. Find the values of a and b. Find also the value of y when $x=0$.
40. If $x+y=c^2+c$ and $x-y=c^2-c$ find the value of $\frac{x}{y}$.

III. Examples Involving Three Unknowns.

Ex.—*Solve :—*

$$6x + 4y - 2z = 5 \quad . \quad . \quad . \quad (1)$$
$$3x - 2y + 4z = -10 \quad . \quad . \quad . \quad (2)$$
$$5x - 2y + 6z = -13 \quad . \quad . \quad . \quad (3)$$

Multiply equation (2) by 2 and take the result along with equation (1).

$$\therefore 6x + 4y - 2z = 5$$
$$\text{and } 6x - 4y + 8z = -20$$
$$\text{Add } \therefore 12x + 6z = -15$$
$$\therefore 4x + 2z = -5 \quad . \quad . \quad . \quad . \quad (4)$$

Next, subtract equation (2) from equation (3)

$$\therefore 2x + 2z = -3 \quad . \quad . \quad . \quad . \quad (5)$$

Solving equations (4) and (5) in the usual way we get

$$x = -1 \text{ and } z = -\tfrac{1}{2}.$$

Substitute for x and z in (say) equation (1)

$$\therefore -6 + 4y + 1 = 5$$
$$\therefore y = 2\tfrac{1}{2}.$$

x	-1
y	$2\frac{1}{2}$
z	$-\frac{1}{2}$

If we examine the method used in the above working, we see that we have eliminated one of the unknowns y, between two pairs of the original equations; namely (1) and (2), and (2) and (3), and so have obtained a pair of simultaneous equations (5) and (6) between x and z. These we know how to solve. It is then easy to find y from any one of the original equations. This process is perfectly general.

EXAMPLES VIII (c).

Solve:—

1. $2x - 3y + z = 1$
   $3x + 2y - z = 3$
   $4x + 5y - z = 7.$
2. $6x - y + 2z = 10$
   $4x + 2y - 3z = -1$
   $5x + 4y - z = 10.$
3. $3x - 18y - 2z = -20$
   $x - 6y + 12z = 6$
   $2x + 8y + 3z = 1.$
4. $4x - 5y + 3z = 2$
   $2x + 3y - 6z = -14$
   $8x + 2y + 5z = 2.$
5. $6x - 4y - 5z = 8$
   $3x + 8y + 10z = -1$
   $9x + 2y + 4z = 11.$
6. $12f - 7g - 25c = 0$
   $10f - 42g + 5c = 0$
   $f - 7g + 4c = 3.$
7. $2a - 5b + 7c = 3$
   $8a + 10b - 9c = 60$
   $15a - 20b + 24c = 9.$
8. $7x - 6y = -1$
   $5y - 4z = 2$
   $3z - 2x = 11.$

CHAPTER IX.

PROBLEMS INVOLVING EQUATIONS OF THE FIRST DEGREE.

I. In this chapter we shall deal with problems which can be solved by the help of equations of the first degree in one or two unknowns. In many problems it will be found that though it is not necessary to employ more than one unknown in the solution, yet the use of two unknowns will make it easier for the pupil to write down the algebraic statements.

Ex. 1.—*Two boxes of chocolates are together worth* 7/6. *One costs twice as much as the other. Find the price of each.*

Let x/- be the price of the cheaper box
$\therefore$ $2x$/- is „ „ „ dearer „
$\therefore$ $3x$/- is the total price of both boxes.
But 7/6 or $7\frac{1}{2}$/- is the price „ „
$\therefore$ $3x = 7\frac{1}{2}$
$\therefore$ $x = 2\frac{1}{2}$.

Hence the cheaper box costs 2/6, and the dearer box costs 5/-.

Ex. 2.—*A husband, wife, and child, weighed themselves at a railway station. The man was three times as heavy as his child and* 3 *stones heavier than his wife. Find the weight of each if together they weighed* 25 *stones.*

Let x stones be the man's weight
$\therefore$ $(x - 3)$ stones is the wife's weight,
and $\frac{1}{3}x$ „ „ child's „
Now their total weight is 25 stones.
$\therefore$ $x + (x - 3) + \frac{1}{3}x = 25$
$\therefore$ $x = 12$.

Hence the man weighed **12 stones**, his wife **9 stones**, and their child **4 stones.**

Ex. 3.—*If* 7 *lb. of tea and* 4 *lb. of coffee cost* £1 17/-, *while* 10 *lb. of tea and* 8 *lb. of coffee cost* £3 2/-, *find the price of tea and of coffee per pound.*

Let the tea cost x/- per lb.
And the coffee cost y/- per lb.
$\therefore$ 7 lb. of tea and 4 lb. of coffee cost $(7x + 4y)$/-
and 10 lb. „ „ 8 lb. „ „ $(10x + 8y)$/-.
But the two lots cost 37/- and 62/- respectively
$\therefore 7x + 4y = 37$
and $10x + 8y = 62$.
Whence $x = 3$ and $y = 4$.
Hence the tea cost **3/- per lb.**
and the coffee cost **4/- per lb.**

Ex. 4.—*A sum of* £4 *is composed of half-crowns and florins. There are* 37 *coins in all. How many are there of each kind?*

Let x = the number of half-crowns
and y = „ „ florins.
$\therefore x + y = 37$ . . . . (1)

Now x half-crowns $= 5x$ sixpences, y florins $= 4y$ sixpences, and £4 = 160 sixpences.

$\therefore 5x + 4y = 160$ . . . (2)

On solving equations (1) and (2) we have $x = 12$ and $y = 25$.
Hence there are **12 half-crowns** and **25 florins.**

Ex. 5.—*A number consists of two digits, whose difference is* 4. *If it be added to the number formed by reversing the digits, the result is* 110. *Find the number, given that the tens' digit is the greater.*

First method. Let x = the tens' digit
and y = the units' digit.

As we get a smaller number by reversing the digits then x must be greater than y (compare 86 and 68).

$\therefore x - y = 4$ . . . . (1)

Now the given number contains $10x + y$ units so that the other number must contain $10y + x$ units

$\therefore (10x + y) + (10y + x) = 110$
$\therefore 11x + 11y = 110$
$\therefore x + y = 10$ . . . . (2)

On solving equations (1) and (2) we have

$$x = 7 \text{ and } y = 3.$$

Hence the number is 73.

Second method. Let x = the units' digit

$\therefore$ $x + 4$ = the tens' digit

$\therefore$ the given number contains $10(x + 4) + x$ units;

i.e. $11x + 40$ units

so that the other number must contain $10x + (x + 4)$ units,

i.e. $11x + 4$ units.

Hence $(11x + 40) + (11x + 4) = 110$

$\therefore$ $x = 3$

$\therefore$ the units' digit is 3 and the tens' digit is 7

$\therefore$ the number is 73.

EXAMPLES IX.

1. A boy is 2 stones heavier than his sister. Together they weigh 12 stones. Find the weight of each.

2. On adding to a vessel twice the quantity of liquid in it, it is found to contain 9 quarts. How much was in it at first?

3. To seven times a number add 3, and the result will be 31. What is the number?

4. A man is three times as heavy as a boy. Together they weigh 16 stones. What is the weight of each?

5. A man is seven times as old as his son. If the difference in their ages is 30 years, how old is each?

6. A man is 25 years older than his son, and the sum of their ages is 49 years. Find the age of each.

7. Think of a number, double it, and add 9 to the result. If doing so gives a number which is five times the original one, find the latter.

8. One box is three times as heavy as another. If 3 lb. be added to the heavier and 11 lb. to the lighter, they will be of equal weight. What does each weigh?

9. One boy has three times as much money as his friend. If the former has 1/4 more than the latter, what does each have?

10. Of an eight weeks' holiday a man spent so much at home, three times as long with a friend, and four times as long on a tour. How long did he spend at home?

11. One stick is three times as long as another. If 11 in. be cut off the first and 3 in. off the second, the remaining portions will be equal in length. How long are the sticks?

12. From two-thirds of a number take 2 and the result will be half of the original number. Find the latter.

13. A box and matches together weigh three times as much as the box. If the matches are 6 drams heavier than the box, find the weight of each.

14. The vertical angle of an isosceles triangle is 30° greater than either of the base ones. Find the angles of the triangle.

15. When a number is doubled and 8 taken from the result, the number thus obtained is two-thirds of the original one. Find the latter.

16. The sum of two numbers is 20 and their difference is 4. Find them.

17. The sum of two numbers is 16 and the larger exceeds twice the smaller by 1. Find the numbers.

18. The difference of two numbers is 9. The larger exceeds twice the smaller by 3. Find the numbers.

19. The sum of two numbers is 30 and their quotient is 9. Find them.

20. The difference of two numbers is 36 and their quotient is 7. Find them.

21. The larger of two numbers increased by twice the smaller is 13. The smaller increased by twice the larger is 14. Find the numbers.

22. Two bottles of lime juice and three bottles of lemon squash cost 13/-. Three bottles of lime juice and one bottle of lemon squash cost 9/-. Find the price of a bottle of each beverage.

23. The result of taking 1 from a number is eight-ninths of that obtained by adding 1 to it. Find the number.

24. A number is such that when 4 is deducted from it, then two-thirds of this result is equal to one-half of that obtained by deducting 2 from the number. Find the number.

25. A man holds 5 shares more in one bank than in another. He sells his holding in the first at £10 a share, and in the second at £12 a share, and so realises £490. How many shares did he hold in each bank?

26. I buy two fountain pens, one plain and one with a gold band, and pay 25/- for them. The one with the band costs half as much again as the other. What did each cost?

27. To one charity a man gave a certain sum, to a second twice as much, and to a third £5 more than to the first. In all he gave away £45. How much did he give to each charity?

28. It is known that the sum of the four angles of any quadrilateral is 360°. If, starting with the smallest, the angles increase by 8°, find the size of each.

29. The sum of three consecutive numbers is 36. Find them. (Let x = the middle number.)

30. If the sum of three consecutive numbers diminished by 15 is equal to half the middle number, find them.

31. The perimeter of a rectangle is 36 in. and the length is twice the breadth. Find the sides.

32. Divide 5/- among 3 boys, so that the first has 1/- more than the second, and the second 6d. more than the third.

33. The sum of the three angles of any triangle is 180°. If one of them is 50° and the remaining angles differ by 12°, find these angles.

34. In a certain provincial theatre a seat in the stalls costs three times as much as a seat in the pit. If 5 seats in the stalls cost 7/- more than 8 in the pit, find the cost of a seat in each place.

35. A man sells an article at 2/- more than he pays for it. By so doing he finds that the sale of 5 of the articles will meet the cost of 7. Find the cost price of the article.

36. One angle of a triangle is twice another and is 20° greater than the third angle. Find all the angles of the triangle.

37. A, B, C, and D are four points on a straight line, such that BC is 6 in. longer than AB, and CD is 6 in. longer than BC. If AD = 30 in. find AB, BC and CD.

38. The smallest angle of a triangle is 2° less than the next in size and 7° less than the largest. Find all the angles.

39. A man walked for a certain distance and then drove for 6 miles. If the distance he walked was a third of his whole journey, how far did he walk?

40. Of three pocket knives purchased, the cheapest cost 6d. less than the next in price, and 1/- less than the dearest. If the total bill was 9/- find what each cost.

41. On a farm there are five times as many acres of wheat as of barley, while the oats cover 20 acres less than the wheat. In all there are 90 acres. Find how many acres are devoted to each crop.

42. A shopman has three articles on his counter together worth 8/6. The dearest is worth twice as much as the next in value and the latter is worth 6d. more than the cheapest. Find the price of each article.

43. A fishing rod is 14 ft. long and is in three pieces which decrease in length by 6 in. Find the length of each piece.

44. A wire is divided into two parts one of which is two-thirds of the other. If one portion is 4 in. longer than the other, find the length of the wire.

45. A cutler is selling two kinds of pocket knives such that two of the dearer cost as much as three of the cheaper. If I buy one of each, I pay 10/- in all, find the cost of each.

46. A dealer bought 12 young pigs and 3 calves paying £51 for them. If a calf cost £2 more than a pig, find the price of each.

47. On a certain day oats were selling at £3 per qr. and barley at £5 per qr. A man sold x qr. of the former and y qr. of the latter for £130. If he had sold y qr. of oats and x qr. of barley he would have received £206. How much of each did he sell?

48. The perimeter of a rectangle is 40 in. If the length were doubled and the breadth halved, the perimeter would be increased by 16 in. Find the dimensions of the rectangle.

49. A caterer bought 20 lb. of tea and 10 lb. of coffee from his grocer and paid £4 for them. The tea cost him 1/- per lb. more than the coffee. Find the price of each per lb.

50. Two shares in one company are worth £3 less than one share in another company, but the value of five shares in the first is £3 greater than one share in the second. What is the value of a share in each company?

51. To 12 men and 8 boys an employer pays £10 a day in wages. The labour of 15 men and 4 boys would cost him £11. Find the daily wages of a man and of a boy.

52. The sum of two numbers is 63. If the greater is increased by 2, the result will be half as much again as the smaller. Find the numbers.

53. There are two numbers whose sum is 125, and such that three-quarters of the one exceeds two-thirds of the other by 13. Find the numbers.

54. £100 was divided between A and B so that after each had spent £6 A had three times as much money as B. Find their shares.

55. A man is 28 years older than his son. Eight years ago the sum of their ages was 36 years. Find the age of each.

56. A bill of £1 19/- was paid in half-crowns and florins, the number of coins being 17. Find how many coins of each kind there were.

57. A bill of £2 was paid in shillings and half-crowns, there being 5 more shillings than half-crowns. How many coins of each kind were there?

58. A man was paid 3/- an hour for ordinary work and 4/- an hour for overtime. One week he worked 56 hours and earned £9. How was his time divided?

59. A "first early" potato is entered in a seedsman's catalogue at 1/6 per stone more than a "main crop" variety. If 2 stones of the first and 5 stones of the second cost 17/-, find the catalogued price of each kind.

60. At a shop one customer buys 8 lb. of apples and 5 lb. of pears for 12/6, while another buys 5 lb. of apples and 9 lb. of pears for 14/8. Find the price of apples and pears per lb.

61. If 5 lb. of tea and 3 lb. of coffee cost 15/10, while 3 lb. of tea and 5 lb. of coffee cost 14/10, find the price of tea and coffee per lb.

62. How many pounds of tea at 3/4 per lb. and at 2/6 per lb. must be mixed together to make a box of 1 cwt. worth £17?

63. Tea costs six times as much as sugar. If 1 cwt. of tea and 1 stone of sugar cost £17 3/-, find the price of 1 lb. of each commodity.

64. If B gives A two-thirds of his money then A will have £36. If, on the other hand, A gives B three-quarters of his money then B will have £36. How much money has each?

65. A rectangular block is 6 in. longer than it is wide, and 1 in. wider than it is deep. The total length of all its edges is 68 in. Find the dimensions of the block.

66. A and B had £50 between them. On A repaying to B a sum of £6 which he had borrowed, then B's money was two-thirds of A's money. How much had each to start with?

67. At a side show a boy pays 1d. for 3 shots with an air-gun and receives back 1d. for every hit he makes. He misses 16 times and finds he has gained 4d. How many hits does he make?

68. A sum of money is divided between a man and a boy. The man receives 10/- more than half of it and the boy 2/6 more than one-fourth of it. What is the sum of money?

69. At a shooting range a boy receives 6d. for a hit but pays 2d. for a miss. After 12 shots he receives 8d. How many times did he hit?

70. A motor car does a journey in 5 hr. which a second requires 6 hr. to do. If the first travels 4 m.p.h. faster than the second, what is the speed of each?

71. A number consists of two digits whose sum is 11. If the digits be reversed the number is increased by 27. Find the number.

72. A number consists of two digits whose sum is 6. If it be added to twice the number formed by reversing the digits, the result is 81. Find the number.

73. A number consists of two digits. If it be added to the number formed by reversing the digits the result is 77, while if it be taken from this number the result is 27. Find the number.

74. By taking a return ticket for a certain railway journey instead of single tickets each way, a man saves 9d. on third-class fare. If a single ticket and a return ticket together cost 5/3, find the cost of each.

75. A pound of fresh butter and a pound of salted butter together cost 5/7. If the price of the former falls 20 per cent. and the price of the latter 25 per cent., the cost will then be 4/4. Find the price of 1 lb. of each kind of butter.

76. One year a man saves £50. Next year his income is increased by 10 per cent. and his expenditure falls 16 per cent. If he then saves £146, find his income for the first year.

77. The sum of three numbers is 75 and the greatest exceeds the least by 9. The difference between the greatest number and the middle one is twice that between the middle number and the least. Find the numbers.

78. A man travels 100 miles in 4 hr., part of the way by train at 40 m.p.h. and the rest by motor at 20 m.p.h. How many miles does he travel each way.

79. Divide £400 between A, B, and C so that C's share may be £70 more than twice A's and £40 less than thrice B's.

80. A dealer buys 476 eggs, x of them at 10 a shilling and the rest at 16 a shilling. By selling all the eggs at 14 a shilling he makes 2/- profit. Find x.

81. P had £12 more than Q. P gave Q one-fourth of his money and then Q had £6 more than P. How much money had each?

82. A man divided £6000 among his three sons. To determine the share of the youngest he added £2410 to three times the share of the eldest and divided the result by 4. The share of the second was £30 more than half the sum of the other two shares. How much did each son get?

CHAPTER X.

GRAPHS I.

Statistical Graphs.

I. On looking up a well-known year book we find the following figures relating to the imports of cotton from India in recent years :—

Year.	1898	1899	1900	1901	1902	1903	1904	1905
Imports in millions of lb.	27	31	35	37	32	81	95	49
Year.	1906	1907	1908	1909	1910	1911	1912	1913
Imports in millions of lb.	71	106	70	75	108	79	57	51

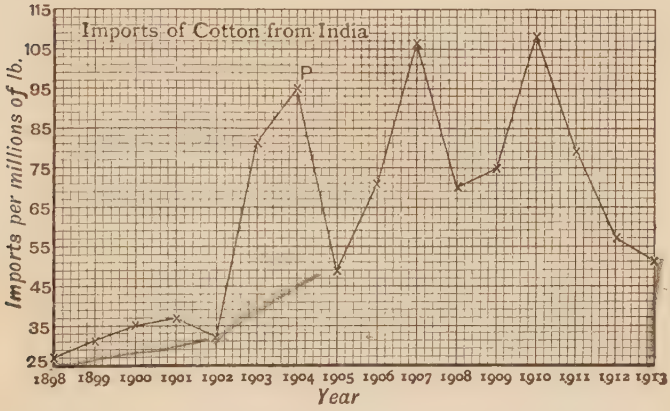
Now it is much easier to grasp the details of a set of statistics such as the above if they are mapped out in the form of a **graph** or diagram drawn on squared paper as shown opposite.

The construction of the diagram needs little explanation.

Two lines or **axes** are drawn on the squared paper, one horizontally and the other vertically. Starting at 1898, $\frac{1}{2}$ in. on the horizontal axis denotes an interval of one year, while on the vertical axis, an interval of $\frac{1}{2}$ in. indicates an import of 20,000,000 lb. The given statistics are then represented by the points marked by crosses. Thus the point P indicates that in the year 1904, 95,000,000 lb. of cotton were imported. Finally to aid the eye in following the rise and fall, the points are joined by straight lines, and the diagram is complete.

The graph shows us at a glance that during the years 1898-1901 the imports increased slowly and fairly steadily, that in

the years 1904, 1907, 1910, they were much higher than usual, that from 1910 to 1913 they fell off rapidly, and so on; but it is to be clearly understood that no new information beyond that given in the table is obtainable from the graph.



We simply get that information presented in the most easily assimilated form.

II. Graphs can, however, be used to amplify our statistics. Thus take the following figures relating to the length of the shortest day in various latitudes:—

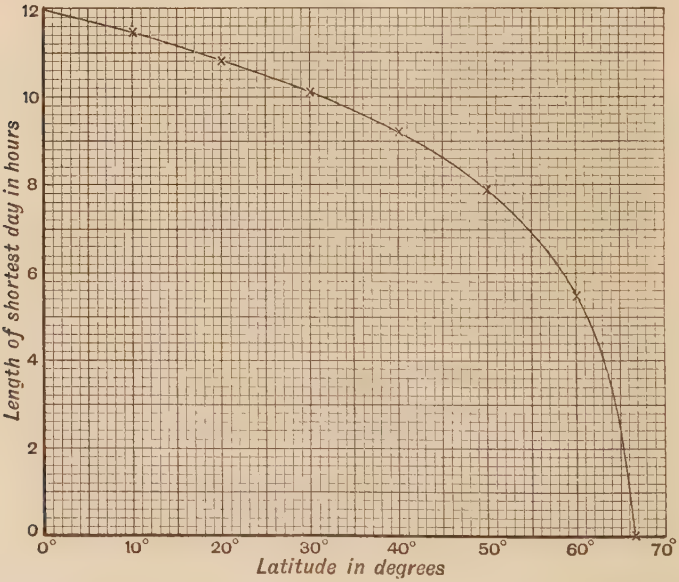
Latitude.	0°	10°	20°	30°	40°	50°	60°	66½°
Length of shortest day in hours.	12·0	11·4	10·8	10·1	9·2	7·9	5·5	0

On plotting the given data we obtain a series of points (marked with crosses) as shown in the Diagram (p. 64).

Now manifestly the length of the shortest day alters continuously from latitude to latitude, and for every latitude intermediate between those given in the table there is some corresponding length of shortest day. To find these pairs of values the proper course is to draw a smooth curve (see Diagram) passing through the plotted points.

Reading off from the curve we find that at latitude 45° the shortest day is **8·6 hr.** while at latitude 56° it has diminished to **6·6 hr.**

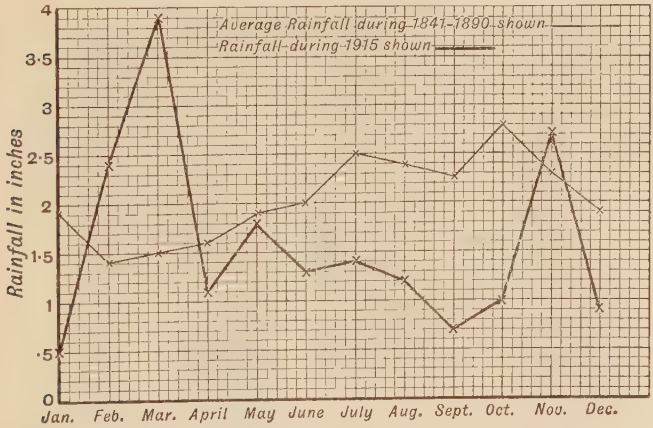
Thus from the table which gives us information only at certain particular latitudes we can construct a chart from which we can estimate the length of the shortest day at all points on the earth's surface from the equator (lat. 0°) to the Arctic circle (lat. 66½°). This process of obtaining values at intermediate points is called *interpolation.*

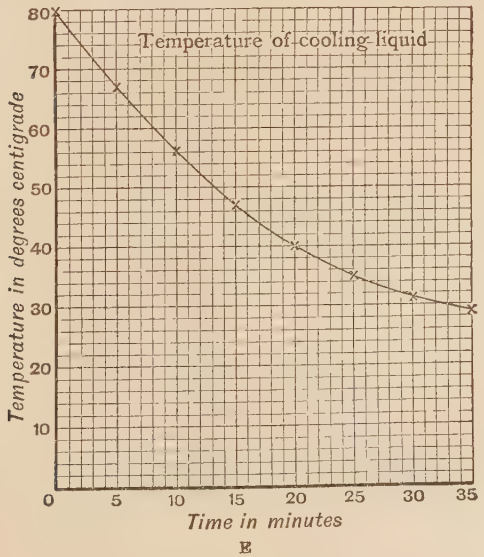


EXAMPLES X (a).

1. The following diagram shows the rainfall at a certain Observatory for each month of the year 1915 and also the average rainfall taken over the years 1841-1890. Read off: (1) the three wettest months of 1915. (2) The three driest months of 1915 and the actual rainfall in April, May, and June, 1915. (3) Would you describe the summer of 1915 as a dry or a wet one? (4) In which month of the year, on the average, does the greatest quantity of rain fall?

(5) On the average, which is the driest period of the year, Spring, Summer, Autumn, or Winter? (6) In which month is the difference between the rainfall in 1915 and the average rainfall (*a*) greatest, (*b*) least? State the differences in inches.





2. A quantity of liquid is heated to a temperature of 80° and is then allowed to cool. The temperature is read off at intervals of five minutes, and the results are recorded on squared paper. A smooth curve is then drawn through the plotted points as shown in the figure on the previous page.

Read off the temperatures after 8, 12, 18, and 22 min. Find also how long is required for the temperatures to fall (*a*) from 60° to 50°, (*b*) from 40° to 30°. Does the temperature fall off more rapidly at the beginning or at the end of the experiment? How does the general shape of the graph indicate this?

3. The time required to roast certain weights of meat are shown in the following table :—

Weight in lb.	1	2	3	4	5	6
Time in min.	40	60	80	100	120	140

Plot these figures and say what you notice about the points you find. Complete the graph, and hence find the time required to cook a $4\frac{1}{2}$ lb. roast, and also what weight of meat would be cooked in an hour and a half.

4. The following figures give the extension E, in tenths of an inch of a spiral spring when various weights W lb. are suspended from one end :—

W	2	4	6	8	10	12
E	3	6	9	12	15	18

Draw a graph from these figures and hence find what weight will be required to give an extension of 1 in.

5. A man buys a gramophone and a number of records. The total cost varies with the number of records according to the following table :—

No. of Records	0	5	10	15	20
Cost in pounds	12	$13\frac{1}{2}$	15	$16\frac{1}{2}$	18

Graph these statistics and find the cost if he were to purchase 12 records, and how many records he would get if he were willing to spend £17 8/- in all.

6. When water flows at a certain speed through pipes of various diameters the quantity delivered per minute is shown below :—

Diameter in in.	0	1	2	3	4	5	6
Quantity in cu. ft.	0	·2	·8	1·8	3·2	5	7·2

Find graphically what quantity would be delivered by a $3\frac{1}{2}$ in. pipe, and also what size of pipe would be required to deliver 4 cu. ft. per min.

7. At a curve on a railway line the outer rail must be raised above the level of the inner. The following table shows the heights H in. necessary on curves of various radii R yd. when the greatest speed allowed is 40 miles per hour :—

R	1000	2000	3000	4000	5000
H	6	3	2	1·5	1·2

Graph these figures and hence find the radius for which a height of 4 in. would just be sufficient.

8. The table given below shows the times required for a carrier to pass along pneumatic tubes of various lengths under given conditions :—

Length in yd.	0	500	1000	1500	2000	2500	3000	3500
Time in min.	0	0·3	0·8	1·5	2·4	3·4	4·4	5·5

Draw a graph showing the relation between the length and the time taken, and hence estimate the time taken to traverse tubes (1) 1700 yd., (2) 2800 yd. long, and the length of tube which could be passed over in 3 min.

9. The following prices of deep well pumps are given in a manufacturer's catalogue :—

Diameter of barrel, d in.	2	3	4	5
Price of pump £ p	17	20	30	47

Plot p and d on squared paper and estimate from your graph the probable cost of pumps of $3\frac{1}{2}$ in. and $4\frac{1}{2}$ in. diameter.

10. In designing roofs the wind pressure is usually taken to be 40 lb. per sq. ft. when the roof is inclined at 40° to the horizontal. The annexed table gives the pressure for other angles :—

Angle . .	10°	20°	30°	40°	50°
Press. in lb. .	9·6	18·4	26·5	33·0	38·0

Find graphically the pressure for an angle of 45°.

The Graphical Solution of Problems.

We shall now explain how graphical methods can be used to solve problems of various types.

Ex. 1.—*A starts at 8 a.m. to cycle from Edinburgh to Glasgow and travels at a uniform rate of 7 m.p.h. B starts at 9 a.m. and follows A at the rate of 9 m.p.h. Find by a graphical construction when and where B will overtake A.*

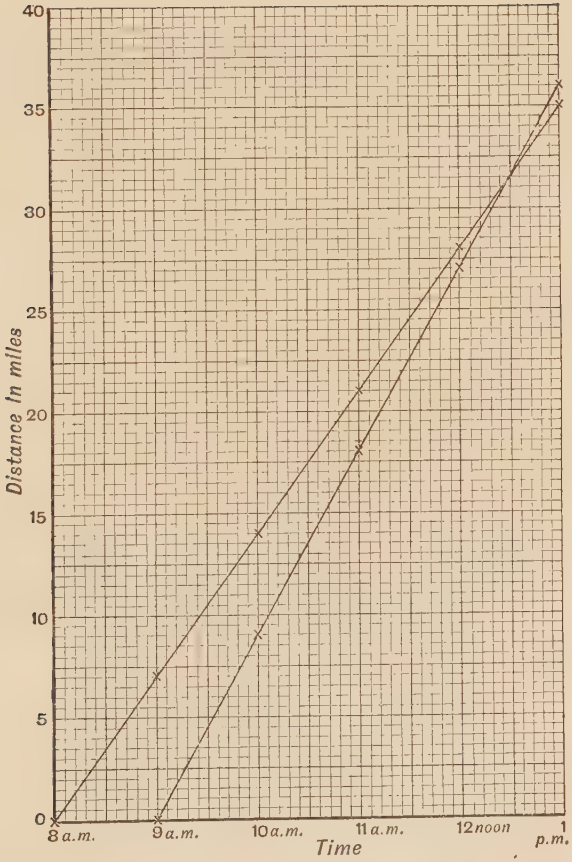
On calculating the respective distances of A and B from Edinburgh at various times we have the following results :—

Time . .	8 a.m.	9 a.m.	10 a.m.	11 a.m.	12 a.m.	1 p.m.
A's distance in miles .	0	7	14	21	28	35
B's distance in miles .		0	9	18	27	36

Clearly B must have passed A before 1 p.m.

Let us then plot on the same diagram, (i) A's distance, (ii) B's distance, against the time.

Both graphs are straight lines, and the point of intersection shows that B must have passed A at 12.30 p.m. and $31\frac{1}{2}$ miles from Edinburgh.

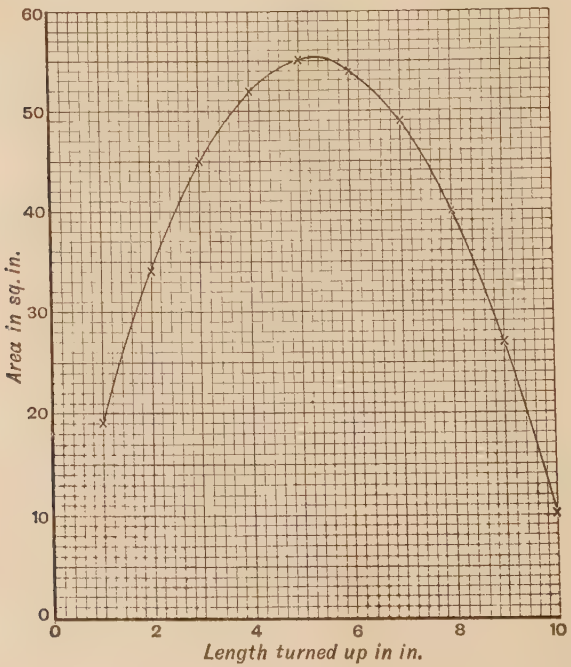

40
35
30
25
20
15
10
5
0
Distance In miles
8 a.m.
9 a.m.
10 a.m.
11 a.m.
12 noon
1 p.m.
Time

Ex. 2.—*A plumber wishes to make a duct for water by bending up the edges of a long strip of sheet lead 21 in. broad into the form of three sides of a rectangle. Find what length he should bend up in order that the area of the section may be 42 sq. in. Show also that it is impossible to make the sectional area as large as 60 sq. in., and find what is the greatest possible area and the corresponding dimensions of the duct.*

The plumber would probably begin by estimating the effect of turning up first 1 in. then 2 in., and so on. Let us follow his example and tabulate our results :—

Length turned up in in. . .	1	2	3	4	5	6	7	8	9	10
Area of section in sq. in. . .	19	34	45	52	55	54	49	40	27	10

Graphing these results



we can now see that the plumber has the choice of turning up either 2·7 in. or 7·8 in. Further, it is at once evident from the diagram that to make the sectional area 60 sq. in. is impossible, for the greatest value it can have is just over 55 sq. in., and the lengths of the side and base of the duct are then $5\frac{1}{4}$ in. and $10\frac{1}{2}$ in. respectively.

Further experience will show us that whenever there are (as in the above case) two solutions to a problem, a purely algebraic method would involve a knowledge of a class of equations called *quadratic* equations. These are fully discussed in Chapters XIII. and XVIII. In particular see page 83 where a problem similar to the above is dealt with.

EXAMPLES X (b).

1. A man desires to buy a gas iron and a length of piping. The iron alone costs 12/- and the piping $4\frac{1}{2}$d. per foot. Fill in the following table giving the total cost for various lengths of piping :—

Length of piping in ft.	0	4	8	12	16
Cost in shillings . .	12				

Graph your results and hence find to the nearest half-foot what length of piping he can obtain if he is willing to spend 17/- in all.

2. To convert weights given in kilograms into pounds we use the relation 1 Kg. = 2·2 lb. Fill in the following table :—

No. of Kg.	0	5	10	15	20	25
No. of lb.	0					

Hence draw a graph which will enable us to read off all weights up to 25 Kg. in lb. Read off the values of 13 Kg. in lb. and of 27 and 41 lb. in Kg.

3. Water stands in two tall jars, in one to a height of 12 in and in the other to a height of 5 in. More water is then poured steadily

into each so that the levels rise at the rate of $\frac{3}{4}$ in. and $1\frac{1}{4}$ in. per sec. respectively. Calculate the height of the water in each at intervals of 4 sec. up to 20 sec. Then graph your results against the time for each vessel in the same diagram. Hence find after what time the levels are the same and the common height at that instant.

4. P leaves Aberdeen for Ballater at noon cycling at the rate of 10 m.p.h. Q follows him at 1.12 P.M. at the rate of 15 m.p.h. Find from a graph when B will overtake A, and at what distance from Aberdeen. Find also how much earlier Q will arrive at Ballater which is 42 miles from Aberdeen.

5. Two cyclists start at noon, one from London at the rate of 12 m.p.h., and the other from Dover, 72 miles distant at 10 m.p.h. Draw graphs representing their motions and hence find at what time, and at what distance from London they will meet, and also at what times they will be 20 miles apart.

6. A starts to walk along a road at noon. He walks at the rate of 4 m.p.h. but rests during alternate hours. B starts from the same place to walk along the same road at 1 P.M. and walks without resting at 3 m.p.h. Taking one inch horizontally to represent an hour, and half-an-inch vertically to represent a mile, draw on the same diagram graphs to represent the progress of each up to 6 P.M. Find from your graphs at what times and places A and B are together and verify your results by calculation.

7. A man can row up a stream at the rate of 2 m.p.h., and down at the rate of 6 m.p.h. Find graphically how far up the river he can row in order that he may return to the starting-point in $2\frac{1}{2}$ hrs. from the time of his departure on the upward journey. [*Hint.*—Draw the graphs representing his up-stream and down-stream motion and find their point of intersection.]

8. A rectangle is such that the sum of two adjacent sides is 5 ft. If x ft. is the length of one side, calculate the area corresponding to the values 0, 1, 2, 3, 4, 5 of x. Plot these results, and hence find the greatest value which the area of the rectangle can have, and the lengths of the sides when it has that value. Find also the lengths of the sides when the area is 5 sq. ft.

9. One side of a rectangle is 2 in. longer than the other. Taking x in. to represent the length of the shorter side, calculate the area corresponding to the values 5, 6, 7, 8, 9, 10 of x. Hence find graphically the lengths of the sides of the area of the rectangle is 75 sq. in.

10. A man calculates that if a tax were put on a certain commodity the consumption would be reduced by one-tenth of the original consumption for every penny of tax per lb. Assuming that 100 lb. would be consumed if no tax were imposed, complete the following table:—

Rate of tax in pence per lb.	0	1	2	3	4	5	6	7	8	9	10
Total duty in pence	0	90	160								

Hence find graphically what rate would yield (i) the maximum duty, (ii) half the maximum duty.

11. A rectangle has an area of 12 sq. in. If the length of one side be x in. calculate the perimeter of the rectangle corresponding to the values, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, 5, 6 of x, and graph your results. Hence find the least perimeter which a rectangle of area 12 sq. in. can have and the corresponding lengths of sides. What is the shape of the rectangle?

12. P is a point on a line AB 6 in. long, complete the following table where the values of AP are given in inches:—

AP	0	1	2	3	4	5	6
$AP^2 + 2PB^2$							

Use the data you have obtained to find graphically the position of P when $AP^2 + 2PB^2$ is (i) 28 sq. in., (ii) a minimum.

CHAPTER XI.

MULTIPLICATION AT SIGHT.

I. On account of the frequency with which they occur it is necessary to be able to write down rapidly simple algebraic products of the type $(ax + b)(cx + d)$.

Consider, for example, the product $(2x + 3)(3x - 4)$.

The full working might be exhibited as follows :—

$$\begin{aligned}(2x + 3)(3x - 4) &= 3x(2x + 3) - 4(2x + 3) \quad . \quad \text{(a)}\\ &= 6x^2 + 9x - 8x - 12 \quad . \quad . \quad \text{(b)}\\ &= 6x^2 + x - 12.\end{aligned}$$

After a little practice steps (a) and (b) should be done mentally and the result written down directly.

Ex.—*Solve the equation* :—

$$(2x + 1)(3x + 1) - (2x - 1)(x + 2) = 4(x + 3)(x - 1) - 15.$$

$$\therefore (2x+1)(3x+1) - (2x-1)(x+2) = 4(x+3)(x-1) - 15.$$

$$\therefore (6x^2 + 5x + 1) - (2x^2 + 3x - 2) = 4(x^2 + 2x - 3) - 15 \quad . \quad \text{(a)}$$

$$\therefore 6x^2 + 5x + 1 - 2x^2 - 3x + 2 = 4x^2 + 8x - 12 - 15.$$

$$\therefore -6x = -30.$$

$$\therefore x = 5.$$

Since a minus sign in front of a bracket affects all the signs inside the bracket it is advisable to retain the brackets, as in line (a) until the multiplication has been performed.

EXAMPLES XI (a).

Write down the following products :—

1. $(x + 1)(x + 2)$.	**6.** $(p - 3)(p - 5)$.	**11.** $(y - 6)(y - 12)$.
2. $(a + 3)(a + 4)$.	**7.** $(p + 3)(p - 5)$.	**12.** $(4 + y)(5 + y)$.
3. $(x + 2)(x + 9)$.	**8.** $(y - 6)(y + 7)$.	**13.** $(7 - p)(4 + p)$.
4. $(x + 8)(x + 4)$.	**9.** $(x - 4)(x + 6)$.	**14.** $(3 - x)(6 - x)$.
5. $(b - 2)(b - 4)$.	**10.** $(x + 7)(x - 10)$.	**15.** $(x + 4y)(x + y)$.

16. $(x - 5y)(x - 2y)$.
17. $(y - 4z)(y + 7z)$.
18. $(2x + 3)(x + 1)$.
19. $(2x - 3)(x + 2)$.
20. $(k - 2)(2k + 1)$.
21. $(y + 3)(3y - 2)$.
22. $(2a + 5)(2a + 3)$.
23. $(3y - 7)(y - 1)$.
24. $(x - 6)\ 5x + 1)$.
25. $(2m + 3)(3m + 2)$.
26. $(5x - 2)(x - 4)$.
27. $(2a - 1)(4a - 1)$.
28. $(7x - 1)(3x + 2)$.
29. $(3y - 8)(2y + 1)$.
30. $(6x + 1)(4x - 3)$.
31. $(2m + 9n)(2m - 5n)$.
32. $(3y - x)(5y - 2x)$.
33. $(3x - 8y)(3x + 2y)$.
34. $(4x + 3a)(4x - 5a)$.
35. $(4x + 3y)(7x - 3y)$.
36. $(3y - 2x)(7y + 3x)$.
37. $(2x - 5y)(5x - 2y)$.
38. $(2pq + 5)(3pq - 2)$.
39. $(2a - bc)(a - 5bc)$.

Simplify :—

40. $(x + 5)(x + 1) + x(x + 6)$.
41. $(x + 4)(x - 3) - x(x + 1)$.
42. $(a - 4)(a - 5) + 10(a - 2)$.
43. $(p - 7)(p - 6) - 3(14 - p)$.
44. $(c + 8)(c - 3) - 5(c - 5)$.
45. $(2x + 3)(2x + 1) - 4x(x + 2)$.
46. $(3x + 5)(x - 2) + (x - 5)(x - 7)$.
47. $(4x + 3)(3x - 1) - (6x + 1)(2x + 1)$.
48. $(x + 3a)(2x - a) - (x - a)(x - 4a)$.

Solve the following equations :—

49. $(x + 1)(x + 3) = (x - 1)(x + 6)$.
50. $(x - 3)(x - 5) = (x + 1)(x - 8)$.
51. $(x + 2)(x - 2) = (x - 3)(x + 5)$.
52. $(2x + 5)(x - 2) = (2x - 7)(x + 6)$.
53. $(3x + 4)(4x - 3) = (6x + 1)(2x - 1)$.
54. $(2x + 3)\ 2x - 3) = (4x - 1)(x + 2)$.
55. $(x + 1)(x + 2) + (x + 3)(x - 5) = x(2x - 1)$.
56. $2(x - 2)(x - 5) - (x + 3)(x - 2) = (x - 4)(x - 5)$.
57. $(2x - 3)(x - 2) - (2x + 1)(x - 4) = x$.
58. $1 - (x + 4)(x + 1) = 5x - (x + 3)(x + 5)$.
59. $2(2x + 7)(x - 1) - 3(x + 1)(x - 8) = (x + 5)(x + 8)$.

II. Three Standard Types.

(A) $$(x + a)^2 = (x + a)(x + a) = x^2 + 2ax + a^2.$$

(B) $$(x - a)^2 = (x - a)(x - a) = x^2 - 2ax + a^2.$$

These results are of the highest importance, and may be stated in words as follows :—

(i) **The square of the sum of two quantities is equal to the sum of their squares plus twice their product.**

(ii) **The square of the difference of two quantities is equal to the sum of their squares minus twice their product.**

For example—

$(2a+5)^2 = (2a)^2 + 2 \,.\, 2a \,.\, 5 + 5^2$
$= 4a^2 + 20a + 25$

$(3a-2b)^2 = (3a)^2 - 2 \,.\, 3a \,.\, 2b + (2b)^2$
$= 9a^2 - 12ab + 4b^2$

(C) $(x+a)(x-a) = x^2 - a^2$.

This rule may be stated as—

The product of the sum and difference of two quantities is equal to the difference of their squares.

Thus $(7a+5b)(7a-5b) = 49a^2 - 25b^2$

EXAMPLES XI (b).

Expand the following squares :—

1. $(x+4)^2$.
2. $(y+1)^2$.
3. $(m+7)^2$.
4. $(p-5)^2$.
5. $(3-m)^2$.
6. $(a-10)^2$.
7. $(x+2y)^2$.
8. $(x-4y)^2$.
9. $(1-3a)^2$.
10. $(1-8b)^2$.
11. $(3a+2)^2$.
12. $(2b-3)^2$.
13. $(4m-5)^2$.
14. $(2x+7)^2$.
15. $(5p-4q)^2$.
16. $(7x-2y)^2$.
17. $(3a+4b)^2$.
18. $(7p-5q)^2$.
19. $(10a+b)^2$.
20. $(2x-5y)^2$.
21. $(x^2-4y^2)^2$.
22. $(2m^2-9n^2)^2$.
23. $(ab+2c)^2$.
24. $(3x-2yz)^2$.

25. Write down the square root of the following expressions :—

(i) $a^2+2ab+b^2$. (ii) $a^2-2ab+b^2$. (iii) $x^2-6xy+9y^2$. (iv) $p^2+4pq+4q^2$. (v) $m^2-10m+25$. (vi) $4c^2-12cd+9d^2$. (vii) $9a^2-12ab+4b^2$. (viii) $25a^2-20a+4$. (ix) $16x^2+24xy+9y^2$.

Write down the following products :—

26. $(a+3)(a-3)$.
27. $(c+5)(c-5)$.
28. $(p+1)(p-1)$.
29. $(7p-10)(7p+10)$.
30. $(4m-3n)(4m+3n)$.
31. $(a+8b)(a-8b)$.
32. $(2p+3q)(2p-3q)$.
33. $(5m-9n)(5m+9n)$.
34. $(10a-3b)(10a+3b)$.
35. $(5x+12y)(5x-12y)$.
36. $(1+11x^2)(1-11x^2)$.
37. $(a^2+2b)(a^2-2b)$
38. $(6-x^2)(6+x^2)$.
39. $(4x-3a^2)(4x+3a^2)$.
40. $(5pq-11r)(5pq+11r)$.

Simplify :—

41. $(3a+2x)^2 - 4x(3a+x)$.
42. $(5p-q)^2 - 6q(2p+q)$.
43. $(x+y)^2 - (x-y)^2$.
44. $(4a+b)^2 - 8a(2a+b) + (a-b)(a+b)$.
45. $5(m-2n)^2 - 2(m-3n)(m+3n)$.

Solve the equations :—

46. $(3x+5)(3x-5)-9(x+1)^2=7x-9$.
47. $2(3x+8)(2x-3)-3(2x-1)^2=1$.
48. $(2x+5)(3x-2)-(2x+3)^2=2(x-3)(x+3)$.
49. $(3x+1)^2-2(x-3)^2=7(x+1)^2$.
50. $(3x+5)(3x-5)-2(2x+1)^2=4(x-2)^2-3(x+1)^2$.

III. Extensions of the Preceding Types. (To be omitted at a first reading.)

Ex. 1.

$$\begin{aligned}(a+b+c)^2&=\{(a+b)+c\}^2\\&=(a+b)^2+2c(a+b)+c^2\\&=a^2+2ab+b^2+2ac+2bc+c^2 \text{ and on re-arranging}\\&=a^2+b^2+c^2+2ab+2ac+2bc.\end{aligned}$$

When the result is written in this last form, it is clear that the required expansion is obtained by adding to the sum of the squares of the individual terms twice each of the partial products formed by taking the terms two at a time. Thus—

$$\begin{aligned}&(2a-3b-4c)^2\\&\quad=(2a)^2+(-3b)^2+(-4c)^2+2.2a(-3b)+2.2a(-4c)+2(-3b)(-4c)\\&\quad=4a^2+9b^2+16c^2-12ab-16ac+24bc\end{aligned}$$

Ex. 2. $(a+b+c)(a+b-c)=\{(a+b)+c\}\{(a+b)-c\}$
$=(a+b)^2-c^2=a^2+2ab+b^2-c^2$.

Ex. 3.

$$\begin{aligned}(x+2y-3z)(x-2y+3z)&=\{x+(2y-3z)\}\{x-(2y-3z)\}\\&=x^2-(2y-3z)^2\\&=x^2-(4y^2-12yz+9z^2)\\&=x^2-4y^2+12yz-9z^2.\end{aligned}$$

EXAMPLES XI (c).

Expand the following squares :—

1. $(a+2b+c)^2$.
2. $(x+y-z)^2$.
3. $(x-y-z)^2$.
4. $(2x+3y+z)^2$.
5. $(2x-y+3z)^2$.
6. $(3x-2y+a)^2$.
7. $(4a+b-2c)^2$.
8. $(5x-6y+z)^2$.
9. $(3x+4y-z)^2$.
10. $(2l+5m+2n)^2$
11. $(3p-q-2r)^2$.
12. $(p+4q-3r)^2$.

Find the following products :—

13. $(a+b+3)(a+b-3)$.
14. $(a-b+5)(a-b-5)$.
15. $(a+2b+c)(a-2b-c)$.
16. $(a-3b-2c)(a+3b+2c)$.
17. $(2x+y-6z)(2x+y+6z)$.
18. $(3x-y+2z)(3x+y-2z)$.

CHAPTER XII.

FACTORISATION OF QUADRATIC EXPRESSIONS.

I. In the last chapter we performed multiplications giving products such as $x^2 + 5x + 6$, $3x^2 + 5x - 2$, $4x^2 + 4x + 1$, etc. These results belong to the general type $ax^2 + bx + c$ and are called **quadratic** expressions. Strictly speaking, any such expression can be factorised, but in the present section we shall deal only with cases where the factors can be found by a simple process of inspection and trial.

A few examples will make the method clear.

Ex. 1.—*Factorise* $x^2 + 11x + 30$.

Since the first term is x^2, the factors must be

$$(x + \quad)(x + \quad).$$

The missing terms must give a product of 30. It only remains therefore to select the proper pair of factors from the following sets :—

(i)	(ii)	(iii)	(iv)
$x + 1$	$x + 2$	$x + 3$	$x + 5$
$x + 30$	$x + 15$	$x + 10$	$x + 6$.

Our choice is to be determined by the fact that the middle term is $11x$. On testing each group in turn we find that (iv) is the correct pair. Hence

$$x^2 + 11x + 30 = (x + 5)(x + 6).$$

Ex. 2.—*Factorise* $x^2 - 9x + 20$.

Since the last term is positive, its factors must have the same sign. They must obviously both be negative to give a negative middle term. The options to be examined are

(i)	(ii)	(iii)
$x - 1$	$x - 2$	$x - 4$
$x - 20$	$x - 10$	$x - 5$.

Calculation of the middle term shows that (iii) is the required pair. Hence

$$x^2 - 9x + 20 = (x - 4)(x - 5).$$

Ex. 3.—*Factorise* $x^2 - 2x - 35$.

Since the last term is negative, the factors must be $(x + \ )(x - \ )$ and the various alternatives are

$$\begin{array}{cccc} x + 35 & x - 35 & x + 7 & x - 7 \\ x - 1 & x + 1 & x - 5 & x + 5. \end{array}$$

By trial the last arrangement gives the correct middle term $- 2x$.

$$\text{Hence } x^2 - 2x - 35 = (x + 5)(x - 7).$$

It should be noted that when, as in the above cases, the coefficient of x^2 is unity, the sum of the missing terms gives the coefficient of x, and the product the term without x.

$$\begin{array}{rlll} \text{Thus in Ex. 1} & 6 + 5 = + 11 & \text{and } 6 \times 5 & = 30 \\ \text{in Ex. 2} & - 4 - 5 = - 9 & \text{and } - 4 \times - 5 & = 20 \\ \text{and in Ex. 3} & 5 - 7 = - 2 & \text{and } 5 \times - 7 & = - 35. \end{array}$$

Ex. 4.—*Factorise* $6x^2 + x - 12$.

Since the coefficient of x^2 is 6, the possible coefficients of x in the factors are 6 and 1, or 3 and 2. The other terms in the factors must give a product of $- 12$, and therefore must have opposite signs, and their numerical values may be either 12 and 1, 6 and 2, or 4 and 3.

At first sight it might seem as if a large number of different possibilities might have to be tested, but it must be remembered that any such factor as $6x - 4$, i.e. $2(3x - 2)$ is impossible, as this would signify that 2 was a factor of the whole expression.

We need therefore try only

$$\begin{array}{cccc} 6x + 1 & 6x - 1 & 3x + 4 & 3x - 4 \\ x - 12 & x + 12 & 2x - 3 & 2x + 3 \end{array}$$

and the last gives the correct middle term $+ x$.

$$\text{Hence } 6x^2 + x - 12 = (3x - 4)(2x + 3).$$

After a little practice the pupil should be able to make the right selection without much difficulty.

Ex. 5.—*Simplify* $\dfrac{x^2 - 5x + 6}{x^2 - 3x + 2}$.

$$\frac{x^2 - 5x + 6}{x^2 - 3x + 2} = \frac{(x - 2)(x - 3)}{(x - 1)(x - 2)}$$

$$= \frac{x - 3}{x - 1}$$

the principle followed in the cancelling being the same as that employed in simplifying the arithmetic product $\dfrac{6 \times 7}{11 \times 6}$.

EXAMPLES XII.

Factorise :—

1. $x^2 + 3x + 2$.
2. $x^2 + 4x + 3$.
3. $x^2 + 5x + 6$.
4. $x^2 - 6x + 8$.
5. $y^2 - 7y + 10$.
6. $x^2 + 11x + 18$.
7. $a^2 - 8a + 7$.
8. $b^2 - 5b + 4$.
9. $x^2 + 3x - 4$.
10. $y^2 + y - 12$.
11. $x^2 - x - 42$.
12. $x^2 - 11x + 10$.
13. $m^2 + 15m + 56$.
14. $a^2 + a - 72$.
15. $x^2 + 5x - 14$.
16. $x^2 - 8x$.
17. $x^2 + 12x$.
18. $a^2 - 18a$.
19. $a^2 + 4a + 4$.
20. $y^2 - 2y - 15$.
21. $m^2 + 2m - 15$.
22. $x^2 - 4x - 77$.
23. $x^2 + 23x + 60$.
24. $x^2 + 13x - 30$.
25. $m^2 - 10m + 24$.
26. $a^2 - 11a + 24$.
27. $p^2 - 7p - 44$.
28. $x^2 - 12x + 36$.
29. $a^2 + 14a + 45$.
30. $a^2 - 4a - 21$.
31. $x(x + 6) + 8$.
32. $x(x - 2) - 63$.
33. $x(x - 13) + 36$.
34. $x^2 - 4(2x + 5)$.
35. $x^2 - 5(2x - 5)$.
36. $m^2 - 9(m + 4)$.
37. $x^2 - 3xy + 2y^2$.
38. $p^2 - 12pq + 35q^2$.
39. $a^2 + 3ab - 40b^2$.
40. $b^2 - 8bc + 16c^2$.
41. $p^2 + 5pq - 50q^2$.
42. $x^2 - 2xy - 48y^2$.
43. $a^2 - 25$.
44. $x^2 - 16$.
45. $x^2 - 9$.
46. $a^2b^2 + ab - 90$.
47. $x^2y^2 + 3xy - 108$.
48. $p^2q^2 - pq - 56$.
49. $2x^2 + 3x + 1$.
50. $2x^2 + 5x + 2$.
51. $2x^2 + 5x + 3$.
52. $4y^2 + 5y + 1$.
53. $3a^2 - 7a + 2$.
54. $5a^2 + 9a - 2$.
55. $2x^2 + 15x + 7$.
56. $5x^2 - 8x + 3$.
57. $3x^2 - 2x - 5$.
58. $6x^2 + 7x - 5$.
59. $3x^2 - x - 2$.
60. $3y^2 + 8y + 5$.
61. $7x^2 + 16x + 4$.
62. $5x^2 + 4xy - y^2$.
63. $6x^2 + 5x - 4$.
64. $8a^2 - 2a - 1$.
65. $4a^2 - 25$.
66. $2x^2 + 7x + 6$.
67. $6x^2 + 13x + 6$.
68. $6x^2 - 5x - 6$.
69. $8x^2 + 3x - 5$.
70. $10x^2 - 7x - 12$.
71. $3x^2 + 11x + 10$.
72. $4x^2 - 5x$.
73. $8x^2 + 6x - 9$.
74. $8x^2 + 38x + 9$.
75. $12x^2 - 16x + 5$.
76. $14a^2 + 29a - 15$.
77. $6x^2 - 11x - 10$.
78. $16a^2 - 49$.
79. $3x^2 - 20x + 33$
80. $4a^2 - 13ab + 10b^2$.
81. $12x^2 - 25xy + 12y^2$.
82. $12x^2 - 7xy - 12y^2$.
83. $12p^2 + 11pq + 2q^2$.
84. $15(x^2 + 1) - 34x$.
85. $8x^2 - 10xy - 7y^2$.
86. $4a^2b^2 - 8ab - 21$.
87. $12a^2b^2 + 37ab + 28$.
88. $4x^4 - 12x^2 + 9$.
89. $2a^4 + a^2 - 21$.
90. $12 + 7x - 10x^2$.
91. $42 - 41x + 5x^2$.
92. $33 - 5x - 2x^2$.
93. $3 - 4t(5 - 8t)$.
94. $4 - 4p^3 + p^6$.
95. $2x^2 - 21xy^2 + 10y^4$.
96. $6x^4 + 7x^2y - 3y^2$.

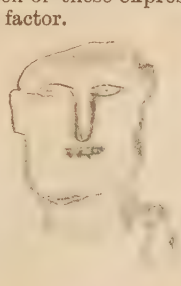
Find a factor common to the following pairs of expressions :—

97. $a^2 - a - 12$ and $2a^2 + 3a - 9$.
98. $2x^2 - 3x - 2$ and $4x^2 + 8x + 3$.
99. $y^2 + y - 6$ and $y^2 + 6y + 9$.

Reduce to their lowest terms :—

100. $\frac{a^2 - 8a + 7}{a^2 - 2a + 1}$.
103. $\frac{x - 3}{x^2 - 2x - 3}$.
106. $\frac{6a^2 + 5a - 21}{3a^2 + 13a + 14}$.
101. $\frac{x^2 + 9x - 10}{x^2 + 12x + 20}$.
104. $\frac{x^2 - 6x}{x^2 - 16x + 60}$.
107. $\frac{5a^2 - 9a - 18}{5a^2 + a - 6}$.
102. $\frac{y^2 + 6y - 27}{y^2 - 6y + 9}$.
105. $\frac{2x^2 + 23x + 11}{x^2 + 10x - 11}$.
108. $\frac{2a^2 - a - 15}{a^2 - 9}$.
109. $\frac{3t^2 + t - 14}{t^2 - 2t}$.
110. $\frac{15x^2 - 14xy - 16y^2}{10x^2 - 31xy + 24y^2}$.
111. $\frac{(a - b)(a - 3b) - 15b^2}{(2a - 3b)(7a + 10b) - 13a^2}$.

112. If $x + 3$ is a factor of $x^2 + ax + 15$, find the value of a.
113. If $x - 2$,, ,, $x^2 - ax + 18$, find a.
114. If $z + 2$,, ,, $z^2 + 12z + b$, find b.
115. If $m - 5$,, ,, $m^2 - 12m + c$, find c.
116. Show by factorising each side that
$(x^2 - 5x - 14)(x^2 + 2x - 3) = (x^2 - 8x + 7)(x^2 + 5x + 6)$.
117. Show that
$(2x^2 + 5x - 3)(4x^2 - 11x - 3) = (8x^2 - 2x - 1)(x^2 - 9)$.
118. Show, by dividing out, that $x - 1$ is one factor of $2x^3 - 5x^2 + x + 2$, and find the others.
119. Show that $x - 2$ is one factor of $x^3 + 2x^2 - 29x + 42$, and find the other factors.
120. Show that $2x - 3$ is a factor of $6x^3 + 13x^2 - 41x + 12$, and find the other factors.
121. Show that $x - 1$ is a common factor of $x^3 - 3x + 2$, and $x^3 + 3x^2 - 4$. Resolve each of these expressions into three factors and find another common factor.



CHAPTER XIII.

QUADRATIC EQUATIONS. I.

I. We are now in a position to discuss the solution of a new class of equations, called *quadratic* equations, of which the following are examples: $x^2 - 7x + 10 = 0$, $3x^2 + 5x = 8$, $4x^2 = 9$, $x(x - 1) = 6$. As we shall see, to every equation of this type there are **two** solutions. By simplification and transposition of terms they all can be reduced to the form $ax^2 + bx + c = 0$, and their solution can be obtained by resolving the left-hand side of the transformed equation into factors as explained in the previous chapter. When these factors cannot be readily found more general methods are required to effect the solution, but a discussion of these is postponed to a later section (*see* Chap. XVIII.).

II. In our subsequent work we shall make use of the following obvious principle:—

If the product of a number of factors is zero, one or other of these factors must be zero.

For example if $abc = 0$, clearly either $a = 0$, or $b = 0$, or $c = 0$.

Ex. 1.—*Solve the equation,* $x^2 + x = 12$.

$$x^2 + x = 12 \quad . \quad . \quad . \quad . \quad (1)$$

Transposing we have

$$x^2 + x - 12 = 0.$$

Factorising the left-hand side

$$(x + 4)(x - 3) = 0.$$

Hence by the principle enunciated above

either $x + 4 = 0$, or $x - 3 = 0$

i.e. either $x = -4$, or $x = 3$.

[Verification.

When $x = -4$. When $x = 3$

In (i) L.H.S. $= (-4)^2 + (-4)$ L.H.S. $= 3^2 + 3$

$= 16 - 4$ $= 9 + 3$

$= 12 =$ R.H.S. $= 12 =$ R.H.S.]

Ex. 2.—*Solve* $15x^2 + 1 = 8(x + 7)$

$$15x^2 + 1 = 8(x + 7).$$

On removing the bracket and transposing

$$15x^2 - 8x - 55 = 0$$

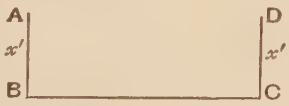
$$\therefore (5x - 11)(3x + 5) = 0$$

$$\therefore 5x - 11 = 0 \quad \text{or} \quad 3x + 5 = 0 \quad . \quad . \quad \text{(A)}$$

$$\therefore x = 2\tfrac{1}{5} \quad \text{or} \quad -1\tfrac{2}{3}.$$

With a little practice line (A) can be omitted.

Ex. 3.—*A plumber wishes to bend a length of sheet lead 6 ft. wide into the form of an open rectangular duct so that the area of the cross section (see diagram below) may be 4 sq. ft. Find the dimensions of the duct.*



Let AB = CD = x ft.

$\therefore$ BC = $(6 - 2x)$ ft. and the area of the section

$= x(6 - 2x)$ sq. ft.

$$\therefore x(6 - 2x) = 4$$

$$\therefore 6x - 2x^2 = 4.$$

Divide throughout by 2 and transpose

$$\therefore x^2 - 3x + 2 = 0$$

whence $x = 1$ or 2.

Hence there are two possible shapes for the duct, one with AB = **1 ft.** and BC = **4 ft.**, and the other with AB = **2 ft.** and BC = **2 ft.**

In this example each of the roots gives an intelligible answer, but it often happens that one of the roots, though it satisfies the equation, does not fit the conditions of the problem.

Ex. 4.—*A manufacturer receives an order for a number of lidless boxes each with a square base and* $5\frac{1}{4}$ *in. high. If the area of the surface is to be* 100 *sq. in., find the side of the base.*

Let the side of the base be x in. We then have

Area of base $= x^2$ sq. in. and the area of one wall $= 5\frac{1}{4}x$ sq. in.

$$\therefore\ x^2 + 4(5\tfrac{1}{4}x) = 100$$
$$\therefore\ x^2 + 21x - 100 = 0$$
whence $x = -25$ or 4.

Now as the length of a side of the base cannot be negative we must neglect the first solution. Hence we decide that the proper length of the side of the square base is **4 in.**

Note.—Whenever it is necessary to reject one of the roots of a quadratic equation, the pupil should state explicitly the reason for doing so.

EXAMPLES.

Solve the following equations ;—

1. $(x - 5)(x - 7) = 0.$
2. $(x - 4)(x - 10) = 0.$
3. $(x - 3)(x + 6) = 0.$
4. $(x + 1)(x + 4) = 0.$
5. $(x + 8)(x - 7) = 0.$
6. $(x + 11)(x + 10) = 0.$
7. $x(x - 3) = 0.$
8. $x(x + 8) = 0.$
9. $(x - a)(x + b) = 0.$
10. $(2x - 1)(2x - 7) = 0.$
11. $(3x + 2)(2x - 9) = 0.$
12. $(4x - 9)(3x + 4) = 0.$
13. $(5x + 11)(2x - 3) = 0.$
14. $2x(3x - 13) = 0.$
15. $(2x - m)(3x - n) = 0.$
16. $(4x + 3m)(3x - 4n) = 0.$
17. $(5x + 6p)(2x + p) = 0.$
18. $(2x + a)^2 = 0.$

Solve ;—

19. $x^2 - 4x + 3 = 0.$
20. $x^2 - 5x + 6 = 0.$
21. $x^2 + x - 2 = 0.$
22. $x^2 - 13x + 40 = 0.$
23. $x^2 - 4x - 45 = 0.$
24. $x^2 + 5x - 14 = 0.$
25. $x^2 - 9x = 0.$
26. $x^2 - 9 = 0.$
27. $x^2 + 5x = 24.$
28. $x^2 + 3x = 4.$
29. $x^2 + 10 = 7x.$
30. $x^2 = 7x.$
31. $x^2 = 9(x - 2).$
32. $8(x - 2) = x^2.$
33. $64 = x(20 - x).$
34. $x(x + 12) = 45.$
35. $x(x + 1) = 72.$
36. $x^2 = 2(5x + 12).$
37. $x^2 = 2(5x - 12).$
38. $x(8 + x) = 48.$
39. $x^2 = 3(5x + 18).$
40. $8(16 - x) = x^2.$
41. $2x^2 - 3x + 1 = 0.$
42. $3x^2 - 5x + 2 = 0.$
43. $2x^2 + x = 6.$
44. $2x^2 + 5x = 7.$
45. $3x^2 - 5x = 2.$
46. $2(x^2 + 2) = 9x.$
47. $3x^2 = 20x + 7.$
48. $5x^2 + 4x = 1.$

49. $5x^2 + 2 = 11x$.
50. $2x^2 + 5 = 7x$.
51. $3x^2 - 8x = 3$.
52. $6x^2 - 7x = 5$.
53. $4x^2 - 20x + 25 = 0$.
54. $6x^2 + x = 12$.
55. $8x^2 - 14x + 3 = 0$.
56. $4x^2 - 19x = 0$.
57. $3x^2 - 5x = 28$.
58. $4x^2 + 28x + 45 = 0$.
59. $8x^2 + 18x = 11$.
60. $6x^2 = 5(x + 5)$.
61. $9x(x + 1) = 10$.
62. $x(3x - 2) = 16$.
63. $9x^2 - 1 = 0$.
64. $4x^2 + 16x + 15 = 0$.
65. $4x(5 - x) = 0$.
66. $4x(x - 2) = 21$.
67. $1 = x(7 - 6x)$.
68. $1 = 3x(2 - 3x)$.
69. $6(3 - x^2) = 23x$.
70. $14 - 3x = 2x^2$.
71. $21x = 10(1 - x^2)$.
72. $x(2x - 13) = 24$.
73. $x - 5 = \frac{36}{x}$.
74. $x^2 - 3ax = 0$.
75. $x^2 + 7ax = 18a^2$.
76. $2x^2 = 9ax + 5a^2$.
77. $\frac{x(x - 1)}{2} = 120$.
78. $(2x + 1)(x + 2) = x(x + 8)$.
79. $(7x - 9)(x + 1) = (4x + 9)(x - 1)$.
80. $(3x - 2)(4x + 3) = (5x + 2)(2x - 1)$.
81. $(x - 2)^2 + (2x - 1)^2 = 9$.
82. $\frac{x}{3} + \frac{3}{x} = 2\frac{1}{2}$.
83. $\frac{x + 2}{6} + \frac{6}{x + 2} = 3\frac{1}{3}$.
84. $(3x - 8)(3x - 5) = 103 - 3(6x - 5)(x - 2)$.

85. Prove that the roots of the equation $x^2 - 19x + 88 = 0$ differ by 3.

86. Prove that one root of the equation $2x^2 - 9x + 9 = 0$ is double the other.

87. Show that the sum of the roots of the equation

$$20x^2 - 9x + 1 = 0 \text{ is } \frac{9}{20}, \text{ and that the product is } \frac{1}{20}.$$

88. If 2 is one root of the equation $x^2 - ax + 24 = 0$, find a and the other root. [*Hint:* find a by substituting 2 for x in the equation.]

89. Two numbers differ by 5, and their product is 84. Find them.

90. Two numbers differ by 4 and the sum of their squares is 58. What are the numbers?

91. The sum of the squares of three consecutive numbers is 29. Find the numbers.

92. The length of a rectangle exceeds the breadth by 9 in., and the area is 52 sq. in. Find the length and breadth.

93. The area of a rectangle is 56 sq. in., and the perimeter is 30 in. Find the sides of the rectangle.

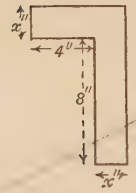
94. Divide a line 12 in. long into two parts such that the sum of their squares may be 80 sq. in.

95. Sixteen times a number exceeds the square of the number by 39. Find the number.

96. The sides of a rectangular sheet of paper are 8 in. and 6 in.

long respectively. If a strip x in. wide is torn off all round, write down expressions for the sides of the remaining rectangle. If its area is 35 sq. in., what is the width of the strip torn off?

97. One side of a rectangular sheet of paper is 4 in. longer than the other. When a strip 2 in. wide is torn off all round the remaining area is 32 sq. in. Find the sides of the original rectangle.



98. Find from the dimensions shown in the annexed sketch the value of x if the total area is 45 sq. in.

99. The hypotenuse of a right-angled triangle is 13 cm., and one of the sides is 7 cm. longer than the other. Find the sides. [Use Pythagoras' Theorem.]

100. The perimeter of a rectangular field is 28 ch. If the diagonal exceeds the length of the shorter side of the field by 4 ch., find the dimensions of the field.

101. A gardener desires to make a gravel path of uniform width round the outside of two adjacent sides of a plot measuring 16 yd. by 13 yd. If he has just sufficient gravel to cover 30 sq. yd., how wide can he make the path?

102. If $y = x - \frac{12}{x}$, find x when $y = 4$.

103. If $y = 6x + \frac{10}{x}$, find x when $y = 17$.

104. When a bullet is fired in a certain manner from a rifle its height above the ground after t sec. is given in ft. by the formula $h = 36t(2\frac{1}{2} - t)$. Find after what times it will be at a height of 36 ft.

CHAPTER XIV.

FURTHER FACTORS.

In an earlier chapter we showed how to obtain the factors of the quadratic expression $ax^2 + bx + c = 0$. We must now consider the problem of finding the factors of other types of algebraic expressions, and for the present we shall deal with the simpler kinds only, leaving the harder cases to be discussed later.

I. When there is a Factor Common to all the Terms of an Expression.

Here the factors can be written down by inspection.
For example :—

(i) $5a+5b=5(a+b)$; (ii) $4a^2x-8a^3=4a^2(x-2a)$.

From the above it is obvious that one factor of the expression is the factor common to the individual terms, while the other is obtained by dividing the expression by this factor.
Thus in

(i) the common factor is 5 and the quotient is $(a+b)$,
(ii) the common factor is $4a^2$ and the quotient is $(x-2a)$.

Before proceeding further, the pupil should note that in factorising *any* expression in which there is a factor common to the various terms this factor should be taken out first.

Thus $$2x^2+10x+12=2(x^2+5x+6)$$
$$=2(x+2)(x+3).$$

The common factor may itself be a compound expression.
Ex.—*Factorise* $(x+y)(a+b)-c(x+y)$.
The common factor being $(x+y)$ we have.

$$(x+y)(a+b)-c(x+y)=(x+y)\{(a+b)-c\}$$
$$=(x+y)(a+b-c).$$

EXAMPLES XIV (a).

Factorise :—

1. $2x+4y$.
2. $3a-6b$.
3. $10x+5$.
4. $pq-p$.
5. $ax-bx$.
6. $7a+14b-21c$.
7. $3ax+12ay$.
8. x^4+2x^3.
9. $2x^3+2x$.
10. $2a^2+4ab-5ac$.
11. $r^2+r^3+r^4$.
12. $4p^2q+8p^2q^2$.
13. $3x^2-15x^2y$.
14. $3x^2+12x+9$.
15. $2x^2-2x-12$.
16. x^3+x^2-6x.
17. $5x^2-35x+60$.
18. $\frac{1}{2}ah+\frac{1}{2}bh$.
19. $\frac{22}{7}r^2+\frac{44}{7}rh$.
20. $12m^2n-32mn^2+20n^3$.
21. $x(a-b)+cx$.
22. $s(x+a)-bs$.
23. $p^2x-p^2(y+z)$.
24. $3ax-3x(b-c)$.
25. $x(p-q)-y(p-q)$.
26. $4m(a+b)-3n(a+b)$.
27. $5b(x+y)-5c(x+y)$
28. $(3x+4)(x+y)-5(3x+4)$.
29. $(a+b)x-(a+b)y+(a+b)z$.
30. $(a+b)^2+b(a+b)$.
31. $(x+y)^2+(x+y)(x-y)$.
32. $(a+2b)(a-b)+(a+2b)(2a+b)$.

Simplify by the help of factors :—

33. $71\times43+29\times43$.
34. $17\times109+17\times43-17\times52$.

Factorise each of the following pairs of expressions and name the common factor :—

35. $2ax+3x^2$, $2a^2+3ax$.
36. $ab+2b$, a^2+4a+4.
37. p^2+4p, p^2+5p+4.
38. $3xy+9y$, $2x^2-6x-36$.

Reduce the following fractions to their lowest terms :—

39. $\frac{2x+2}{3x+3}$.
40. $\frac{5x-40}{x^2-8x}$.
41. $\frac{5a^2-15a}{4a-12}$.
42. $\frac{3x^2-30x+48}{3x^2-6x}$.
43. $\frac{3m^2-5m-2}{6mn+2n}$.
44. $\frac{pqr+qrs}{qrs+qrt}$.

II. Factorisation by Grouping. (To be omitted at a first reading.)

Sometimes by grouping the terms of an expression in a certain manner the presence of a common factor is revealed. Only practice and experience will enable us to decide how to group the various terms. If one method of grouping will not work we must simply try another.

Ex. 1.—*Factorise* $p^2+2p+pq+2q$.

Grouping in pairs as shown by the curved lines

$$\begin{aligned} \overbrace{p^2+2p}+\overbrace{pq+2q} &= (p^2+2p)+(pq+2q) \\ &= p(p+2)+q(p+2) \\ &= (p+2)(p+q). \end{aligned}$$

Ex. 2.—*Factorise* $ax - by + ay - bx$.

$$\begin{aligned} ax - by + ay - bx &= (ax + ay) - (by + bx) \\ &= a(x + y) - b(y + x) \\ &= (x + y)(a - b). \end{aligned}$$

Ex. 3.—*Factorise* $mn + 12 - 3n - 4m$.

$$\begin{aligned} mn + 12 - 3n - 4m &= (mn - 3n) + (12 - 4m) \\ &= n(m - 3) + 4(3 - m) \\ &= n(m - 3) - 4(m - 3). \\ &= (m - 3)(n - 4). \end{aligned}$$

Note.—The pupil is strongly advised to check his answer to any question on factorisation by multiplying the factors mentally to see whether their product gives the original expression.

EXAMPLES XIV (b).

Factorise :—

1. $ab + 2a + 3b + 6$.
2. $ac + ad + bc + bd$.
3. $ab - 2b - a + 2$.
4. $3ax + a - 12x - 4$.
5. $3x^2 - 3xy + 7x - 7y$.
6. $x^2 - bx - ax + ab$.
7. $px^2 - qx^2 + py^2 - qy^2$.
8. $pq - q^2 - pr + qr$.
9. $a^3 - a^2 + a - 1$.
10. $m^2 - mn + m - n$.
11. $m^2 - mn - m + n$.
12. $ab - cd + bc - ad$.
13. $am - 2an + 2bn - bm$.
14. $2pq + rs - 2qr - ps$.
15. $ab + x^2 - ax - bx$.
16. $6l^2 + 2m - 9l^3 - 3lm$.
17. $a^3 - 2a^2 + a - 2$.
18. $a^5 - 4 + a^3 - 4a^2$.
19. $4x^2y^2 + xy + 2x^3 + 2y^3$.
20. $l^2m^2 - 2 - 2l^2 + m^2$.
21. $abx^2 - a^2x - ab + b^2x$.
22. $y^2 + 5c^2 - cy - 5cy$.
23. $ax + ay + az - bx - by - bz$.
24. $(a + b)(x + y) - cx - cy$.
25. $(2p + q)(a - b) + 2p + q$.
26. $3m + 3n + (m + n)^2$.
27. $4m - 4n - (m - n)^2$.
28. $2ax + 2bx - (a + b)^2$.
29. $(p + 2q)^2 - 5p - 10q$.
30. $y^2 - (3 - a)y - 3a$.
31. $(x + y)(x - y) - 4x - 4y$.
32. $(x + 2y)(x - 2y) - 4x + 8y$.
33. $x^2 - 3x + 2 + a(x - 1)$.
34. $x(y + z - a) - z(x - y + a)$.

Factorise the following pairs of expressions and state the common factor :—

35. $x^2 - 2xy$, $px + 2qx - 4qy - 2py$.
36. $x^2 - 9xy + 14y^2$, $xy + 4yz - 2xz - 2y^2$.
37. $x^4y + 4 - 2x^2 - 2x^2y$, $3x^3 - 6x$.

III. The Difference of Two Squares.

We have already seen (Chap. XI.), that

$$x^2 - a^2 = (x + a)(x - a).$$

Hence it follows that any expression which can be written as the difference of two squares can be factorised according to the following rule :—

The difference of the squares of two numbers is equal to the product of the sum and difference of the numbers.

Ex. 1.—*Factorise* $x^2 - 36y^2$.

$$\begin{aligned} x^2 - 36y^2 &= x^2 - (6y)^2 \\ &= (x + 6y)(x - 6y). \end{aligned}$$

Ex. 2.—*Factorise* $(2a + 3b)^2 - (a - 2b)^2$.

$$\begin{aligned} (2a+3b)^2 - (a-2b)^2 &= \{(2a+3b) + (a-2b)\}\{(2a+3b) - (a-2b)\} \\ &= (2a+3b+a-2b)(2a+3b-a+2b) \\ &= (3a+b)(a+5b). \end{aligned}$$

Ex. 3.—*Factorise* $x^2 - 2yz - y^2 - z^2$.

Here we group the last three terms. Thus

$$\begin{aligned} x^2 - 2yz - y^2 - z^2 &= x^2 - (y^2 + 2yz + z^2) \\ &= x^2 - (y + z)^2 \\ &= \{x + (y + z)\}\{x - (y + z)\} \\ &= (x + y + z)(x - y - z). \end{aligned}$$

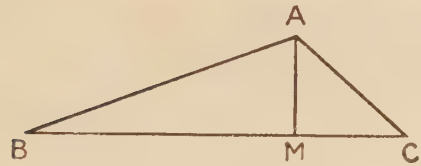
Ex. 4.—*Factorise* $x^2 - 2x - 2y - y^2$.

Grouping as indicated by the curved lines :—

$$\begin{aligned} \overbrace{x^2 - 2x} - \overbrace{2y - y^2} &= (x^2 - y^2) - (2x + 2y) \\ &= (x + y)(x - y) - 2(x + y) \\ &= (x + y)(x - y - 2). \end{aligned}$$

The above rule can frequently be used in arithmetical problems, more especially in those dealing with the measurement of areas, in order to simplify the calculation.

Ex. 5.—*ABC is a triangle in which* $AB = 3{\cdot}7$ *in.;* $BC = 5{\cdot}1$ *in.;* $CA = 2$ *in.; AM is drawn perpendicular to BC. Find the lengths of BM and CM.*



Let BM = x in., MC = y in., AM = p in.

Then $x^2 + p^2 = 3{\cdot}7^2$

$y^2 + p^2 = 2^2$

$\therefore x^2 - y^2 = (3{\cdot}7)^2 - 2^2$

$(x + y)(x - y) = 5{\cdot}7 \times 1{\cdot}7.$

But $x + y = 5{\cdot}1$ . . . . . (1)

$$\therefore x - y = \frac{5{\cdot}7 \times 1{\cdot}7}{5{\cdot}1}$$

$= 1{\cdot}9$ . . . . . (2)

From (1) and (2) $x = 3{\cdot}5$, $y = 1{\cdot}6$, i.e. BM = 3·5 in., CM = 1·6 in.

EXAMPLES XIV (c).

Factorise :—

1. $x^2 - 4$.
2. $x^2 - 9$.
3. $y^2 - 25$.
4. $y^2 - 49$.
5. $36 - a^2$.
6. $1 - m^2$.
7. $p^2 - 81$.
8. $4a^2 - 9$.
9. $9x^2 - 1$.
10. $100 - m^2$.
11. $16p^2 - 25$.
12. $x^2y^2 - 36$.
13. $121b^2 - 9c^2$.
14. $16 - a^2b^2$.
15. $100x^2 - 49y^2$.
16. $a^4 - 9$.
17. $36 - a^4b^2$.
18. $x^4 - y^4$.
19. $16m^4 - 81$.
20. $a^4b^4 - 16$.
21. $1 - 16x^4y^4$.
22. $3a^2 - 27$.
23. $5 - 5x^2$.
24. $a^2x - b^2x$.
25. $a^2b - 4b^3$.
26. $2x^2 - 50y^2$.
27. $243 - 3x^4$.
28. $(a + b)^2 - c^2$.
29. $(a - b)^2 - c^2$.
30. $9 - (x + y)^2$.
31. $x^2 - (y - z)^2$.
32. $(x + 5)^2 - 9$.
33. $25 - (4 - x)^2$.
34. $(3x + 1)^2 - 81$.
35. $49 - (7 - x)^2$.
36. $4(a + b)^2 - c^2$.
37. $9(x - 2y)^2 - 4z^2$.
38. $100 - 121(p - q)^2$.
39. $(3a + b)^2 - (2a + 3b)^2$.
40. $(2p - q)^2 - (p + q)^2$.
41. $(4a + 7b)^2 - (3a - 5b)^2$.
42. $(11x - 8y)^2 - (3x - 4y)^2$.
43. $25a^2 - (3a + 7b)^2$.
44. $(x^2 + 2)^2 - 9x^2$.
45. $(x^2 + 2x)^2 - (3x + 6)^2$.

Apply the rule for the difference of two squares to the simplification of the following :—

46. $(51)^2 - (49)^2$.
47. $(5{\cdot}7)^2 - (4{\cdot}3)^2$.
48. $(850)^2 - (750)^2$.
49. $(19{\cdot}8)^2 - (9{\cdot}2)^2$.

50. A square of side 5·7 in. is cut out of a square sheet of tin of side 14·3 in. Find the area left.

51. The hypotenuse of a right-angled triangle is 37 in. and one side is 35 in. Find the other side.

52. If $s = \frac{v^2 - u^2}{2f}$ find s when $v = 73$, $u = 27$, $f = 2$.

53. If $\mathrm{Y} = 3\mathrm{X}^2$ and $y = 3x^2$ prove $\frac{\mathrm{Y} - y}{\mathrm{X} - x} = 3(\mathrm{X} + x)$.

Reduce to their lowest terms :—

54. $\frac{xy + 2y}{x^2 - 4}$.

55. $\frac{x^2 - x - 12}{2x^2 - 32}$.

56. $\frac{a^2 - 144}{5a - 60}$.

57. $\frac{4m^2 - 9}{4m^2 - 12m + 9}$.

58. If $x^2 - y^2 = 143$ and $x + y = 13$ find x and y.

59. ABC is a triangle, AN is drawn perpendicular to BC. In each of the following cases calculate BN and CN. Hence determine AN and the area of the triangle.

(i) AB = 20 in. BC = 21 in. CA = 13 in.
(ii) AB = 17 in. BC = 28 in. CA = 25 in.
(iii) AB = 20 in. BC = 11 in. CA = 13 in.

60. Prove that $(x^2 + x - 12)(x^2 - x - 12) = (x^2 - 9)(x^2 - 16)$.

61. Factorise $(x^2 + 3x + 1)^2 - 1$. Hence prove

$$x(x + 1)(x + 2)(x + 3) + 1 = (x^2 + 3x + 1)^2.$$

62. Prove that $x(x + 2)(x + 4)(x + 6) + 16 = (x^2 + 6x + 4)^2$.

[The following examples should be omitted at a first reading.]
Factorise :—

63. $a^2 + 2ab + b^2 - c^2$.
64. $a^2 - 2a + 1 - b^2$.
65. $4x^2 + 4xy + y^2 - z^2$.
66. $x^2 - y^2 + 2yz - z^2$.
67. $9a^2 - b^2 - 2bc - c^2$.
68. $16a^2 - 6pq - 9p^2 - q^2$.
69. $x^2 - 25y^2 - z^2 + 10yz$.
70. $25a^2 - c^2 + 10c + 1$.
71. $36 - 12ab - 4a^2 - 9b^2$.
72. $4mn + l^2 - 4m^2 - n^2$.
73. $a^2 - 1 - b^2 + 2b$.
74. $s^2 - c^2 - 6as + 9a^2$.
75. $p^2 + q^2 - r^2 - s^2 + 2(pq + rs)$.
76. $k^2 - m^2 + l^2 - n^2 - (2kl - mn)$.
77. $x^2 - y^2 - 9(x - y)$.
78. $x^2 - y^2 - 9x - 9y$.
79. $p^2 - 3p - q^2 - 3q$.
80. $x(x - 4) - y(y - 4)$.
81. $a(a - 1) - b(b - 1)$.
82. $ax + ay - (x^2 - y^2)$.
83. $5m + 5n - m^2 + n^2$.
84. $a^4 - b^4 + a^2x^2 - b^2x^2$.
85. $16 - m^2 - (4 + m)^2$.

86. If $\mathrm{Y} = a + b\mathrm{X} + c\mathrm{X}^2$ and $y = a + bx + cx^2$, prove

$$\frac{\mathrm{Y} - y}{\mathrm{X} - x} = b + c(\mathrm{X} + x)$$

87. Show that $x^4 + x^2y^2 + y^4 = (x^2 + y^2)^2 - x^2y^2$. Hence factorise $x^4 + x^2y^2 + y^4$.

88. If $a^2 = b^2 + c^2 - 2bc(2x - 1)$, prove $x = \dfrac{b^2 + c^2 + 2bc - a^2}{4bc}$.

Resolve the numerator of the right-hand side into factors, and show that if $2s = a + b + c$, then $x = \dfrac{s(s - a)}{bc}$.

IV. The Sum or Difference of Two Cubes.

(To be omitted at a first reading.)

The pupil may easily verify that if we divide $a^3 + b^3$ by $a + b$, the quotient is $a^2 - ab + b^2$, while if we divide $a^3 - b^3$ by $a - b$, the quotient is $a^2 + ab + b^2$.

We therefore have the following results :—

(i) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
(ii) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

These identities may be stated in words as follows :—

(i) *The sum of the cubes of two numbers is equal to the product of two factors, the first of which is the sum of the numbers while the second is the sum of their squares diminished by their product.*

(ii) *The difference of the cubes of two numbers is equal to the product of two factors, the first of which is the difference of the numbers, while the second is the sum of their squares increased by their product.*

It may be of assistance to the pupil to note that in both cases there is only **one** minus sign on the right-hand side.

Ex. 1.—*Factorise* $27x^3 + 64$.

$$27x^3 + 64 = (3x)^3 + (4)^3$$
$$= (3x + 4)(9x^2 - 12x + 16).$$

Ex. 2.—*Factorise* $125a^3 - 1$.

$$125a^3 - 1 = (5a)^3 - (1)^3$$
$$= (5a - 1)(25a^2 + 5a + 1).$$

EXAMPLES XIV (d).

Factorise :—

1. $x^3 - 1$.
2. $a^3 + 1$.
3. $1 - a^3$.
4. $1 + x^3$.
5. $x^3 + 8$.
6. $8 - x^3$.
7. $y^3 - 27$.
8. $64 - x^3$.
9. $8m^3 + 1$.
10. $x^3 - 125y^3$.
11. $27x^3 - a^3$.
12. $8x^3y^3 + 27$.

13. $1000 - m^3$.
14. $8l^3 - 125m^3$.
15. $216 - p^3q^3$.
16. $343x^3 - 27y^3$.
17. $64m^3 + 125n^3$.
18. $729 - 8a^3$.
19. $3n^3 + 81$.
20. $4x^3 + 4$.
21. $m^4 - m$.
22. $5c^3 - 40d^3$.
23. $2 - 250a^3$.
24. $a^3x^2 - b^3x^2$.
25. $a^6 + b^6$.
26. $a^6 - 8b^6$.
27. $64p^6 - 27$.
28. $(a+b)x^3 + (a+b)y^3$.
29. $a^3 + b^3 + 3ab(a+b)$.
30. $a^3 - b^3 - 3ab(a-b)$.

Simplify :—

31. $\dfrac{x^3-1}{x-1} + \dfrac{x^3+1}{x+1} - \dfrac{x^4-1}{x^2-1}$.
32. $\dfrac{4x^2+4x}{4(x+1)} + \dfrac{ax^3-a}{ax^2+ax+a}$.

Reduce to their lowest terms :—

33. $\dfrac{x^3+1}{(x+1)^3}$.
34. $\dfrac{x^3-8}{x^2-4}$.
35. $\dfrac{(x-3)^3}{x^3-27}$.
36. $\dfrac{2x^2-3x-2}{8x^3+1}$.
37. $\dfrac{x^4+27x}{x^3+12x^2+27x}$.
38. $\dfrac{3a^3-24}{3a^2+9a-30}$.

Revision Exercise on Factors.

Factorise the following :—

1. $2x^2 + 8x$.
2. $2x^2 + 8x + 6$.
3. $ab^3 - a^3b$.
4. $3x^2 - 300$.
5. $x^2 - 16x + 39$.
6. $3 + 11x - 4x^2$.
7. $(a+2x)m - (a+2x)n$.
8. $a^4 - 13a^2 + 36$.
9. $a^2(3+x) - b^2(3+x)$.
10. $x^5 - x$.
11. $a^2b^2 - 11ab + 30$.
12. $7 - 6x - x^2$.
13. $3p^2 - 7p - 10$.
14. $x(x-5) + 4$.
15. $x^3 + 11x^2 + 18x$.
16. $10x^2 - xy - 3y^2$.
17. $\dfrac{a^2}{4} - 100$.
18. $(m+3n)^2 - 1$.
19. $2x^2 + 26x - 96$.
20. $(a+b)^2 + 5(a+b)$.
21. $15a^2 + 7a - 36$.
22. $32 - 4x - 3x^2$.
23. $(5p-6q)^2 + q(5p-6q)$.
24. $l^2 - (l-m)^2$.
25. $(a+2x)^2 - (a-2x)^2$.
26. $(x+1)(x+2) + (x+1)(y-2)$.
27. $6x^2 - 5xy - 56y^2$.
28. $mn + 1 - m - n$.
29. $(2x^2+3x+1)^2 - (x^2+4x+5)^2$.
30. $9x^4 - 82x^2y^2 + 9y^4$.
31. $(2x+3)(3x+4) + 5(3x+4)$.
32. $m(4x+3y) - 8x - 6y$.
33. $4x^3 - b^2x$.
34. $p(q-1) - q^2 + 1$.
35. $15(m^2-1) - 72m$.
36. $(p^2-2p)^2 - 9$.
37. $(x^2-2)^2 - x^2$.
38. $x^3 - x^2 - x + 1$.
39. $(a+b)(x^2+4x) + 4a + 4b$.
40. $9x^2 - (y+z)^2$.
41. $15 - x(11-2x)$.
42. $x^4 - 81y^4$.
43. $7x^3 + 189c^3$.
44. $(a+b)^3 - 4(a+b)$.
45. $a(a-9) - x(x-9)$.
46. $(ax+by)^2 + (bx-ay)^2$.
47. $x^2 - x(a-2b) - 2ab$.
48. $9x^2 - y^2 - 15x + 5y$.
49. $pq^3 - 64p$.
50. $9a^2 - c^2 + 24ab + 16b^2$.
51. $(a+b)^2 + 2(a+b) + 1$.
52. $4b^2 - 80b + 364$.

53. $(x^2+x)^2-8(x^2+x)+12$.
54. $a^3+a^2b+ab^2+b^3$.
55. $a^2(1+b)-b^2(1+a)$.
56. $a(p+q-r)-q(a-p+r)$.
57. $p^5+p^2q^3-p^3q^2-q^5$.
58. $a^2-4a-(a-4)^2$.
59. $a^3-3a^2b+3ab^2-b^3$.
60. $a(x^2-b^2)-b(x^2-a^2)$.
61. $x^2-4a^2-2ab-bx$.
62. $(2x+3y)^2-(x-5y)^2$.
63. $25(a-b)^2-9(a-c)^2$.
64. $(a^2-a)^2-14(a^2-a)+24$.
65. $(x+1)(x+2)+(x+1)(x+3)+2(2x+5)$.
66. $16(3x-2y)^2-(x+3y)^2$.
67. $(x^2+8xy+6y^2)^2-36y^4$.
68. $(m+2n)^3-9m-18n$.
69. $(x+2)(x^2-12x+35)+(x^2-3x-10)(8-x)$.
70. $a(a+2b)+b(b+2c)+c(c+2a)$.

Simplify:—

71. $\dfrac{a^3-b^3}{a-b}+\dfrac{a^3+b^3}{a+b}$.

72. $\dfrac{x^2+4x+3}{3x+9}+\dfrac{x^2+2x+1}{4(x+1)}-\dfrac{7x^2-7}{12(x-1)}$

73. Prove that the difference of the squares of two consecutive even numbers is equal to four times the odd number between them. [Let the numbers be $2x$ and $2x+2$.]

74. Factorise $(x^2+3x-4)^2-(x^2-3x-4)^2$, and find the quotient when this expression is divided by $6x+12$.

75. If $a^2-16b^2=12$, and $a-4b=2$, find a and b.

76. Find the square root of $(x^2+3x+2)(x^2+4x+3)(x^2+5x+6)$.

77. Simplify $\dfrac{(6x^2+7x-3)(6x^2+x-2)}{(4x^2+4x-3)(9x^2+3x-2)}$.

78. Factorise $(a+1)(a+2)(a+3)-2a-2$.

79. Find the quotient on dividing $(3x^2+x-4)^2-(x^2-2x+1)^2$ by $2x^2+3x-5$, and factorise the result.

80. If $x+5$ is a factor of $x^2+ax-45$, find a.

81. Factorise (i) a^4-15a^2-16, (ii) $(a-b)^2+(ab+1)^2$, and name a common factor.

82. Factorise (i) x^3+2x^2+2x+1, (ii) x^6-1, and name the common factors.

83. Show that $x-3$ is a factor of $x^3-x^2-69x+189$, and find the other factors.

CHAPTER XV.

MULTIPLICATION AND DIVISION OF FRACTIONS.

I. The pupil is already familiar with the rules and processes required in dealing with arithmetical fractions. The following examples sufficiently illustrate their application to the operations of multiplication and division of algebraic fractions.

Ex. 1.—*Simplify* $\frac{3x^2}{5y^2} \times \frac{10y}{7x^3}$.

Following the same plan as in arithmetic we divide $3x^2$ and $7x^2$ by the common factor x^2, and $10y$ and $5y^2$ by the common factor $5y$. Hence

$$\frac{3x^2}{5y^2} \times \frac{10y}{7x^3} = \frac{\cancel{3x^2}^{\,3}}{\cancel{5y^2}_{\,y}} \times \frac{\cancel{10y}^{\,2}}{\cancel{7x^3}_{\,7x}} = \frac{6}{7xy}.$$

Ex. 2.—*Simplify* $\frac{xy - x^2}{x^2 + 4xy} \times \frac{x^2 + 5xy + 4y^2}{x^2 - 5xy + 6y^2} \div \frac{x^2 - y^2}{x - 3y}$.

$$\text{Expression} = \frac{\cancel{x}(y - x)}{\cancel{x}(x + \cancel{4y})} \times \frac{(x + y)(x + 4y)}{(x - 2y)(x - \cancel{3y})} \times \frac{x - \cancel{3y}}{(x + y)(x - y)}$$

$$= \frac{(y - x)}{(x - 2y)(x - y)}$$

$$= \frac{-(x - y)}{(x - 2y)(x - y)}, \text{ since } (y - x) = -(x - y)$$

$$= -\frac{1}{x - 2y}.$$

EXAMPLES XV.

Simplify:—

1. $\frac{x^3}{y^2} \times \frac{y}{x^2}$.
2. $\frac{a^5}{b^3} \times \frac{b^4}{a^7}$.
3. $\frac{a^2}{b} \div \frac{a}{b^2}$.
4. $\frac{p^3}{8q} \div \frac{4p}{q}$.
5. $\frac{3x^2}{4y} \times \frac{5y}{6x}$.
6. $\frac{8b^2}{9x^2} \times \frac{3x}{4ab}$.
7. $\frac{2a^2}{7b} \div \frac{5a}{14b^2}$.
8. $\frac{5a^2b}{4c^2} \times \frac{8bc}{15a}$.
9. $\frac{2a^5b}{5xy} \div \frac{8a^3b^2}{25x^2y}$.

10. $\frac{m^2n}{p^2q} \times \frac{2pq^2}{9xy} \times \frac{pxy}{mn^2}$.

11. $\frac{6a^2}{5bc} \times \frac{10b^2}{9ac} \div \frac{3ab}{4c^2}$.

12. $\frac{x(x-7)}{3(x+7)} \times \frac{x+7}{7x}$.

13. $\frac{x^2-4}{2x} \times \frac{4xy}{x+2}$.

14. $\frac{3x+12y}{x+2y} \div \frac{5x+20y}{2x+4y}$.

15. $\frac{x^2-4x+3}{5x} \times \frac{10x^2}{x^2-5x+6}$.

16. $\frac{x^2+3x+2}{x+3} \times \frac{x^2+3x}{x^2+2x+1}$.

17. $\frac{ab-b^2}{a} \times \frac{5ab}{(a-b)^2}$.

18. $\frac{a^2-9}{a^2+3a} \div \frac{a^2+a-12}{a^2+4a}$.

19. $\frac{3p^2q-3pq^2}{5pr-10qr} \div \frac{6p^2-6pq}{p-2q}$.

20. $\frac{x+7}{x-6} \times \frac{2x+10}{3x+21} \div \frac{5x+25}{x^2-6x}$.

21. $\frac{m^2+4m-21}{m+4} \times \frac{m^2-16}{m^2+9m+14} \div \frac{m-4}{m+2}$.

22. $\frac{x^2-1}{x^2+5x+6} \times \frac{2x-8}{x^2+5x+4} \times \frac{x^2+7x+12}{x^2+4x-5}$.

23. $\frac{2x^2+9x+7}{x-2} \times \frac{x^2-4x+4}{2x^2+3x-14}$.

24. $\frac{3x^2-x-2}{3x^2+5x+2} \div \frac{x^2-4x+3}{x^2+4x+3}$.

25. $\frac{5p^2+17p-12}{p^2+8p+16} \times \frac{p^2-2p-24}{5p^2-18p+9}$.

26. $\frac{a^2+a-6}{2a^2+2} \times \frac{a^2-a-6}{a^2+2a-8} \times \frac{a^2+4a}{a^2-9}$.

27. $\frac{8x^2+22x+15}{6x^2-x-1} \times \frac{(x-1)^2}{2x^2-x-6} \div \frac{4x^2+x-5}{3x^2-5x-2}$.

28. $\frac{a^3-ab^2}{b^2c-bc^2} \times \frac{b^2-c^2}{a^4-b^4}$.

29. $\frac{(a+c)^2-b^2}{(a+b)^2-c^2} \div \frac{ab-b^2+bc}{a^2+ab-ac}$.

30. $\frac{x^2-(x-3y)^2}{y} \times \frac{5}{4x^2-9y^2}$.

31. $\frac{9-a^2}{2a^2+6a+4} \times \frac{(a+1)^2}{2a^2-a-15} \times \frac{4a^2+10a}{3+a}$.

32. $\frac{x^3-27y^3}{3y} \div \frac{2x^2+6xy+18y^2}{x^2+4xy+3y^2} \times \frac{6x}{x^2-9y^2}$.

CHAPTER XVI.

THE LOWEST COMMON MULTIPLE.

If we take the expressions x^2y and xy^2 we see at once that x^2y^2 is a multiple of both expressions. There are an unlimited number of other multiples such as x^3y^3, x^4y^4, x^2y^5, and so on, but all these are themselves multiples of x^2y^2. This last expression is called the lowest common multiple (L.C.M.) of x^2y and xy^2.

The method of finding the L.C.M. will be grasped readily from a few examples.

Ex. 1.—*Find the L.C.M. of* a, a^2, a^3.

The L.C.M. is clearly a^3, the highest power of a which occurs, for no lower power will contain a^3.

Ex. 2.—*Find the L.C.M. of* $3a^2b$, $6a^3b^4$, $9a^5b^6c$.

The L.C.M. of 3, 6, 9 is 18.
„ „ „ a^2, a^3, a^5 is a^5.
„ „ „ b, b^4, b^6 is b^6.

As c is the only power of c which occurs, the required L.C.M. is $18a^5b^6c$.

Ex. 3.—*Find the L.C.M. of* $3x^2 - 3xy$, $x^2 - 2xy + y^2$, and $x^2 - 6xy + 5y^2$.

We must first factorise the given expressions :—

$$3x^2 - 3xy = 3x(x - y); \; x^2 - 2xy + y^2 = (x - y)^2;$$
$$x^2 - 6xy + 5y^2 = (x - y)(x - 5y).$$

By the method explained above, the required L.C.M. is $3x(x - y)^2(x - 5y)$.

EXAMPLES XVI.

Find the L.C.M. of :—

1. a, a^2.
2. $2a$, $3a^2$.
3. ab, bc.
4. x^2y, xy^3.
5. $4x^2$, $6x^3$.
6. $8a$, $10ab^2$.
7. $2x$, $3y$, $5z$.
8. $4y^3$, $10y^5$, $20y^4$.
9. $2a^2$, $3ab$, $4a^3$.
10. $3a^3b$, $5a^2b^2$, $4abc^2$.
11. $2x$, $3(x-1)$.
12. $6(x-2)$, $8(x-3)$.
13. $x(x-a)$, $(x-a)(x-b)$.
14. $(x-2)(x-5)^2$, $x^2(x-5)$.
15. x^2-3x+2, x^2-5x+6.
16. $m^2+2m-24$, $m^2+12m+36$.
17. $4x^2-4y^2$, $6x-6y$.
18. a^2-b^2, $(a+b)^2$.
19. $6a^2-6ab$, $9a^2-18ab+9b^2$.
20. $x^2+13x+40$, $5x^2+40x$.
21. $3x^2-7x+2$, $3x^2-10x+3$.
22. $3x^2+4xy$, $9x^2-16y^2$.
23. $10p^2q+20pq^2$, $p^2+pq-2q^2$.
24. $m^2-8mn+16n^2$, m^2-16n^2, m^2-4mn.
25. $2x^2-5x+2$, $(x-2)^3$, $4x^2-4x+1$.
26. $a^2-17a+60$, a^3-10a^2+25a, ab^2-12b^2.
27. $3m^3-3m^2n$, $m^2-5mn+4n^2$, $4mn-16n^2$.
28. $5a^2-5x^2$, $4(a+x)^2$, $6a^2+6x^2$.
29. $8x^3-8a^2x$, $(x+a)^3$, $12x^2+24ax+12a^2$.
30. $6x^2-18xy-24y^2$, $8x^2-40xy-48y^2$, $x^3-10x^2y+24xy^2$

CHAPTER XVII.

ADDITION AND SUBTRACTION OF FRACTIONS.

Before the operations of addition and subtraction of algebraic fractions can be performed it is necessary first to reduce the fractions to a common denominator, by finding the L.C.M. of the given denominators.

Ex. 1.—*Express as a single fraction* $\frac{x}{4a^2} + \frac{y}{2ab} - \frac{z}{6b^2}$.

The L.C.M. of the denominators being $12a^2b^2$ we have

$$\frac{x}{4a^2} + \frac{y}{2ab} - \frac{z}{6b^2}$$

$$= \frac{3b^2x}{12a^2b^2} + \frac{6aby}{12a^2b^2} - \frac{2a^2z}{12a^2b^2}$$

$$= \frac{3b^2x + 6aby - 2a^2z}{12a^2b^2}.$$

Ex. 2.—*Simplify* $\frac{2}{3x - 9} - \frac{4}{x^2 - 9}$.

In order to find the L.C.M. of the denominators we must resolve them into their factors. Then

$$\frac{2}{3x - 9} - \frac{4}{x^2 - 9} = \frac{2}{3(x - 3)} - \frac{4}{(x + 3)(x - 3)}$$

$$= \frac{2(x + 3) - 12}{3(x + 3)(x - 3)}$$

[Since $3(x-3)$ is contained in the L.C.M. $(x+3)$ times, and $(x+3)(x-3)$ is contained in it 3 times.]

$$= \frac{2(x-3)}{3(x+3)(x-3)}$$

$$= \frac{2}{3(x+3)}.$$

Ex. 3.—*Simplify* $\dfrac{x+2}{2x^2+x-3} - \dfrac{x+1}{2x^2+11x+12}$.

Expression $= \dfrac{x+2}{(x-1)(2x+3)} - \dfrac{x+1}{(2x+3)(x+4)}$

$$= \frac{(x+2)(x+4)-(x+1)(x-1)}{(x-1)(x+4)(2x+3)}$$

$$= \frac{(x^2+6x+8)-(x^2-1)}{(x-1)(x+4)(2x+3)}$$

$$= \frac{6x+9}{(x-1)(x+4)(2x+3)}$$

$$= \frac{3(2x+3)}{(x-1)(x+4)(2x+3)}$$

$$= \frac{3}{(x-1)(x+4)}.$$

Ex. 4.—*Simplify* $\dfrac{x^2}{a^2-x^2} + \dfrac{x}{x-a}$.

$$\text{Expression} = \frac{x^2}{(a+x)(a-x)} + \frac{x}{x-a}.$$

Since the denominator of the first fraction contains the factor $(a-x)$, and since $x-a=-(a-x)$ it is convenient to write the second fraction with $(a-x)$ as denominator.

Now just as $\dfrac{6}{-3} = -\dfrac{6}{3}$ so $\dfrac{x}{x-a} = \dfrac{x}{-(a-x)} = -\dfrac{x}{x-a}$

$$\therefore \text{ the given expression} = \frac{x^2}{(a+x)(a-x)} - \frac{x}{a-x}$$

$$= \frac{x^2-x(a+x)}{(a+x)(a-x)}$$

$$= -\frac{ax}{(a+x)(a-x)}.$$

EXAMPLES XVII.

Reduce to a single fraction and simplify :—

1. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}$.
2. $\frac{2x}{3}-\frac{x}{6}+\frac{x}{2}$.
3. $\frac{3x}{4}-\frac{5x}{7}$.
4. $\frac{2a}{15}+\frac{b}{3}+\frac{a}{5}$.
5. $2p-\frac{q}{4}+\frac{p}{6}-\frac{q}{9}$.
6. $\frac{x+1}{2}+\frac{2x+1}{3}$.
7. $4-\frac{x+8}{2}$.
8. $\frac{2m-3}{6}-\frac{m-1}{8}$.
9. $\frac{2a+5}{8}-\frac{a-1}{4}$.
10. $1-\frac{x+3}{4}+\frac{2x-1}{7}$.
11. $\frac{x-y}{3}+\frac{x+y}{2}$.
12. $\frac{a-3x}{4}+\frac{a+2x}{6}$.
13. $\frac{x-3y}{5}-\frac{2x-3y}{10}$.
14. $\frac{2a+b}{3}-\frac{a-3b}{4}$.
15. $\frac{1}{2x}+\frac{1}{3x}$.
16. $\frac{4a}{5x}-\frac{3a}{10x}$.
17. $\frac{2}{y}-\frac{3}{4y}$.
18. $2-\frac{3}{a}+\frac{1}{a^2}$.
19. $\frac{3}{x^2}-\frac{4}{xy}+\frac{2}{y^2}$.
20. $\frac{5a}{2b}-\frac{2a^2}{b^2}$.
21. $\frac{3}{ab}+\frac{4}{bc}$.
22. $\frac{p-1}{5x}-\frac{p-2}{6x}$.
23. $\frac{1-p}{p}-\frac{1-2p}{2p}-\frac{1-3p}{3p}$.
24. $\frac{2x^2-3}{4x^3}-\frac{x-4}{2x}$.
25. $\frac{2x+1}{3x}-\frac{3x+2}{5x}-\frac{1}{15}$.
26. $\frac{b-c}{bc}+\frac{c-a}{ca}+\frac{a-b}{ab}$.
27. $\frac{2(p-q)}{3p^2q}-\frac{p+q}{2pq^2}$.
28. $\frac{2}{x+1}+\frac{1}{x}$.
29. $\frac{3}{x+1}-\frac{2}{x+2}$.
30. $\frac{4}{2x+1}-\frac{2}{x-3}$.
31. $\frac{a}{a-b}-\frac{b}{a+b}$.
32. $\frac{1}{x-1}+\frac{2}{x+1}-\frac{3x-1}{(x-1)(x+1)}$.
33. $\frac{2a+b}{(a-b)^2}-\frac{2}{a-b}$.
34. $\frac{a}{x+a}-\frac{b}{x+b}$.
35. $\frac{3a}{a-3b}-\frac{5a}{a-5b}$.
36. $\frac{a^2}{(a-1)^2}-\frac{a}{a-1}$.
37. $\frac{5}{(x-4)(x+1)}-\frac{3}{(x-4)(x-1)}$.
38. $\frac{5}{xy+2y^2}+\frac{10}{x^2+2xy}$.
39. $\frac{7}{x^2+x-12}-\frac{1}{x^2-5x+6}$.
40. $\frac{3}{a-b}-\frac{2a-5b}{a^2-3ab+2b^2}$.
41. $\frac{1}{x^2-5x+6}-\frac{1}{x^2-3x+2}$.
42. $\frac{2}{x^2-4x+3}-\frac{1}{x^2-3x+2}$.

43. $\dfrac{1}{a^2-3a+2}-\dfrac{4}{a^2+2a-3}+\dfrac{5}{a^2+a-6}$.

44. $\dfrac{1}{2x+3y}+\dfrac{2}{2x-3y}-\dfrac{2x+9y}{4x^2-9y^2}$.

45. $\dfrac{1}{x^2-11x+30}-\dfrac{10}{x^2-2x-24}+\dfrac{9}{x^2-x-20}$.

46. $\dfrac{1}{3(x-3)}+\dfrac{x}{x^2-9}-\dfrac{1}{2(x+3)}$.

47. $\dfrac{2}{x^2-8x+15}+\dfrac{2}{x^2-4x+3}-\dfrac{4}{x^2-6x+5}$.

48. $\dfrac{12a-7}{16a^2-1}-\dfrac{5}{4a+1}+\dfrac{3}{4a-1}$.

49. $\dfrac{6x}{9x^2-4}-\dfrac{1}{3x-2}-\dfrac{1}{3x+2}$.

50. $\dfrac{1}{a-4}+\dfrac{2}{a-2}-\dfrac{2}{a^2-6a+8}$.

51. $\dfrac{9}{a^2-a-20}-\dfrac{2}{a^2-8a+15}-\dfrac{7}{a^2+a-12}$.

52. $\dfrac{x-3}{x^2+x-6}+\dfrac{1}{2x^2-3x-2}+\dfrac{6}{2x^2+7x+3}$.

53. $\dfrac{1-x}{2+x}+\dfrac{2-x}{1+x}+2$.

54. $\dfrac{1}{x-1}+\dfrac{2}{x-2}-\dfrac{3}{x-3}$.

55. $\dfrac{4x-6y}{x^2-3xy+2y^2}-\dfrac{3}{x-y}-\dfrac{1}{x-2y}$.

56. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}$.

57. $\dfrac{1}{x+1}+\dfrac{2}{x^2+3}+\dfrac{8}{(x^2-1)(x^2+3)}$.

58. $\left\{\dfrac{1}{(x-1)(x-2)}+\dfrac{1}{(x-2)(x-3)}\right\}\times\left\{x+\dfrac{3}{x-4}\right\}$.

59. $\dfrac{x-\dfrac{2}{x}-1}{x-\dfrac{6}{x}+1}$.

60. $\dfrac{\dfrac{a}{b}+2+\dfrac{b}{a}}{\dfrac{1}{a}+\dfrac{1}{b}}$.

61. $\dfrac{3x+4}{x-1}+\dfrac{7}{(2x-3)(x-1)}-\dfrac{6x-5}{2x-3}$.

62. $\dfrac{x}{x^2-y^2}+\dfrac{y}{(x+y)^2}-\dfrac{1}{x+y}$.

63. $\dfrac{1}{x+y}-\dfrac{x-y}{(x+y)^2}-\dfrac{2xy}{(x+y)^3}$.

64. $\dfrac{\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}}{x+y-1}$.

65. $\dfrac{1+\frac{1}{x-1}}{1-\frac{1}{x+1}}$.

66. $\dfrac{\frac{1}{a^2}-\frac{1}{4b^2}}{\frac{1}{2b}+\frac{1}{a}}$.

67. $\dfrac{3}{4(a-b)}-\dfrac{5}{8(b-a)}$.

68. $\dfrac{4}{(a+2)(a-2)}+\dfrac{3}{2-a}$.

69. $\dfrac{2}{x-1}+\dfrac{4}{1-x^2}$.

70. $\dfrac{5}{(x-5)(x-10)}-\dfrac{3}{5-x}-\dfrac{4}{x-10}$.

71. $\dfrac{1}{p-2}+\dfrac{4}{1-p}+\dfrac{3p}{p^2-3p+2}$.

72. $\dfrac{1}{x^2-x-2}+\dfrac{2}{1-x^2}+\dfrac{1}{x^2+x-2}$.

73. $\dfrac{1}{2(x-2)}-\dfrac{x-6}{x^2-9x+18}+\dfrac{x-7}{2(x^2-11x+28)}$.

74. $\dfrac{m^2-7m+12}{m^2-8m+15}-\dfrac{m^2-6m+5}{m^2-7m+6}$.

75. $\left(\dfrac{a-b}{a+b}+1\right)\left(\dfrac{a+b}{a-b}-1\right)\left(\dfrac{a}{b}-\dfrac{b}{a}\right)$.

76. $\dfrac{\frac{x+2}{x+1}-\frac{1}{x+3}}{\frac{1}{x+1}+\frac{x+2}{x+3}}$.

77. $\dfrac{2a-x}{2a+x}-\dfrac{4a^2+x^2}{4a^2-x^2}+\dfrac{4ax^3}{16a^4-x^4}$.

78. $\dfrac{\frac{x+4}{7}-\frac{x+4}{x+5}}{\frac{x-2}{4}+\frac{x-2}{x}}$.

79. $\dfrac{x^2-y^2}{y^2-xy}-\dfrac{x^2-y^2}{xy-x^2}+\dfrac{x}{y}$.

CHAPTER XVIII.

QUADRATIC EQUATIONS. II.

I. We have already seen how to solve a quadratic equation in cases where the quadratic expression obtained by taking all the terms to one side can be resolved into simple factors. Sometimes, however, the finding of these factors is a matter of considerable difficulty, and it is therefore necessary to explain a more general method which will enable us to solve any quadratic equation.

Let us consider the equation

$$x^2 = 16.$$

Transposing and factorising we have

$$(x-4)(x+4) = 0$$
$$\therefore x-4 = 0 \text{ or } x+4 = 0$$
$$\therefore x = 4 \text{ or } -4.$$

We may write this $x = \pm 4$.

Similarly it can be shown that for any equation of the type

$$x^2 = a^2$$

the solution is $x = \pm a$ . . . . (1)

Next consider a more complex case

$$(x+2)^2 = 36.$$

Here $x+2 = \pm 6$ by (1)

$$\therefore x = -2+6 \text{ or } -2-6$$
$$= 4 \text{ or } -8.$$

We shall now proceed to show how any quadratic equation can be solved by first reducing it to the above form.

II. In the chapter on "Multiplication at Sight" (see page 75) we have seen that

$$(x+a)^2 = x^2 + 2ax + a^2.$$

Hence, in order to complete the square of the expression x^2+2ax we add a^2, i.e., the square of half the coefficient of x. For example, to complete the square of x^2+12x we add 6^2, i.e. 36, obtaining $x^2+12x+36$ or $(x+6)^2$. Again to complete the square of x^2-20x we add $(-10)^2$, i.e. 100, obtaining $x^2-20x+100$ or $(x-10)^2$.

Ex. 1.—*Solve* $x^2+8x=65$.

If we add to each side the square of half the coefficient of x, viz. 16, the left-hand side will be a perfect square.

For we have
$$\begin{aligned} x^2+8x+16 &= 65+16 \\ \therefore (x+4)^2 &= 81 \\ \therefore x+4 &= \pm 9 \\ \therefore x &= -4+9 \text{ or } -4-9 \\ &= 5 \text{ or } -13. \end{aligned}$$

Ex. 2.—*Solve, correct to* 2 *decimal places* $(x+1)(x-1)=4x$.

Multiplying out and rearranging, we have
$$x^2-4x=1.$$

Completing the square of the left-hand side
$$\begin{aligned} x^2-4x+4 &= 1+4 \\ \therefore (x-2)^2 &= 5 \\ \therefore x-2 &= \pm\sqrt{5} \\ \therefore x &= 2+\sqrt{5} \text{ or } 2-\sqrt{5} \\ &= 2+2{\cdot}236\ \ldots \text{ or } 2-2{\cdot}236\ \ldots \\ &= 4{\cdot}24 \text{ or } -0{\cdot}24 \text{ correct to 2 decimal places.} \end{aligned}$$

Before applying the method of "completing the square" to the solution of a quadratic equation the coefficient of x^2 should be made unity.

Ex. 3.—*Solve* $x=3-2x^2$.

Rearranging so as to have the term in x^2 positive
$$2x^2+x=3.$$
Dividing by 2
$$x^2+\frac{x}{2}=\frac{3}{2}.$$
Completing the square
$$x^2+\frac{x}{2}+\left(\frac{1}{4}\right)^2=\frac{3}{2}+\frac{1}{16}.$$
$$\therefore \left(x+\frac{1}{4}\right)^2=\frac{25}{16}$$

$$\therefore x + \frac{1}{4} = \pm\frac{5}{4}$$

$$\therefore x = -\frac{1}{4} + \frac{5}{4} \text{ or } -\frac{1}{4} - \frac{5}{4}.$$

$$= 1 \text{ or } -\frac{3}{2}.$$

Ex. 4.—*Solve* $\frac{3}{x-1} - \frac{1}{x+3} = \frac{5}{8}$.

Multiply both sides of the given equation by $8(x-1)(x+3)$.

$$\therefore 24(x+3) - 8(x-1) = 5(x-1)(x+3)$$

whence

$$5x^2 - 6x = 95$$

$$\therefore x^2 - \frac{6}{5}x = 19$$

$$\therefore x^2 - \frac{6}{5}x + \left(\frac{3}{5}\right)^2 = 19 + \frac{9}{25}$$

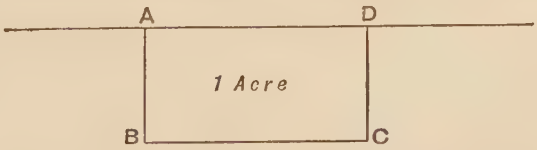
$$\therefore \left(x - \frac{3}{5}\right)^2 = \frac{484}{25}$$

$$\therefore x - \frac{3}{5} = \pm\frac{22}{5}$$

$$\therefore x = \frac{3}{5} + \frac{22}{5} \text{ or } \frac{3}{5} - \frac{22}{5}$$

$$= 5 \text{ or } -3\tfrac{4}{5}$$

Ex. 5.—*A man wishes to fence off an acre of land in the form of a rectangle, using an existing fence as one boundary He possesses* 176 *yd. of fencing. Find whether this is sufficient ?*



We shall work in chains to render the numbers smaller.

Let AB = x ch.
$\therefore$ CD = x ch.
and BC = $(8 - 2x)$ ch. [$\because$ 176 yd. = 8 ch.]

$$\text{Now 1 ac.} = 10 \text{ sq. ch.}$$
$$\therefore x(8 - 2x) = 10$$
$$\therefore 8x - 2x^2 = 10$$
$$\therefore x^2 - 4x = -5$$
$$\therefore x^2 - 4x + 4 = -5 + 4$$
$$\therefore (x - 2)^2 = -1.$$

Now it is impossible to find a real number such that its square is equal to a negative number. Hence we must conclude that it is impossible to enclose a field of the required size with the fencing in the man's possession.

III. Solution of the General Quadratic Equation

$$ax^2 + bx + c = 0.$$

Here $ax^2 + bx = -c$.

Dividing by a, $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

Completing the square,

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$
$$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$
$$\therefore x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$
$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Since any quadratic equation can be written in the form $ax^2 + bx + c = 0$, we have in this example obtained a general formula for the roots of any such equation in terms of the coefficients of x^2 and x, and of the constant term. We can therefore apply our result to solve any quadratic equation merely by substituting the appropriate values for a, b, c.

Ex. 1.—*Solve* $15x^2 - 3x = 1$.

Writing the equation in the proper form,

$$\text{we have } 15x^2 - 3x - 1 = 0.$$
$$\text{Now } a = 15,\ b = -3,\ c = -1.$$

$$\therefore x = \frac{3 \pm \sqrt{9+60}}{30}$$

$$= \frac{3 \pm \sqrt{69}}{30}$$

$$= \frac{3 + 8{\cdot}306}{30} \text{ or } \frac{3 - 8{\cdot}306}{30}$$

$$= 0{\cdot}38 \text{ or } -0{\cdot}18 \text{ correct to 2 decimal places.}$$

Ex. 2.—*A piece of wire* 20 *in. long is cut into two portions, and each portion is then bent into the form of a square. If the sum of the areas of the squares is* 14 *sq. in., how is the wire divided?*

Let the side of one square be x in.

$\therefore$ perimeter of this square $= 4x$ in.

$\therefore$ perimeter of the other square $= (20 - 4x)$ in.

$\therefore$ side of the second square $= (5 - x)$ in.

$$\therefore x^2 + (5 - x)^2 = 14$$

$$\therefore 2x^2 - 10x + 11 = 0$$

$$\therefore x = \frac{10 \pm \sqrt{100 - 88}}{4}$$

$$= \frac{10 \pm \sqrt{12}}{4}$$

$$= 3{\cdot}37 \text{ or } 1{\cdot}63.$$

When $x = 3{\cdot}37$ the portions are 13·5 in. and 6·5 in.

When $x = 1{\cdot}63$ the portions are 6·5 in. and 13·5 in.

$\therefore$ the wire is divided into two parts whose lengths are **13·5 in.** and **6·5 in.**

EXAMPLES XVIII.

[In the following exercises whenever the value of a square root cannot be determined exactly, the roots should be found correct to two decimal places.]

Solve:—

1. $x^2 = 144$.
2. $x^2 = 400$.
3. $x^2 = \frac{36}{49}$.
4. $x^2 = 6\frac{1}{4}$.
5. $(x - 2)^2 = 25$.
6. $(x + 4)^2 = 100$.
7. $(x - 5)^2 = 1$.
8. $(x + 3\frac{1}{2})^2 = 9$.
9. $(x - 8)^2 = \frac{1}{4}$.
10. $(x + 1)^2 = \frac{16}{81}$.
11. $(x - 5)^2 = 2$.
12. $(x + 2)^2 = 3$.
13. $(x - a)^2 = b^2$.
14. $(x + 2b)^2 = a^2$.
15. $(x - p)^2 = \frac{q^2}{4}$.

Complete the following squares :—

16. x^2+4x.
17. x^2-10x.
18. x^2-16x.
19. x^2+20x.
20. x^2+5x.
21. x^2-7x.
22. x^2-x.
23. $x^2+\frac{2}{3}x$.
24. $x^2-\frac{4}{5}x$.
25. $x^2+\frac{1}{3}x$.
26. $x^2-\frac{3}{4}x$.
27. x^2+3ax.

Solve by completing the square (or by the formula, if desired) :—

28. $x^2+2x=3$.
29. $x^2-10x=39$.
30. $x^2-12x=-11$.
31. $x^2-18x+45=0$.
32. $x^2+2x-80=0$.
33. $168=x^2-2x$.
34. $x^2=34x+240$.
35. $x^2+3x=40$.
36. $x^2-5x=84$.
37. $60-7x=x^2$.
38. $4(x+11)=x^2-1$.
39. $x^2+2x=5$.
40. $x^2-4x=7$.
41. $x^2-10x=3$.
42. $x^2-6x+4=0$.
43. $x^2+6x+1=0$.
44. $x^2+8x+14=0$.
45. $x^2=2(1-2x)$.
46. $x(x-10)=19$.
47. $(x+4)(x-4)=12x$.
48. $14x+1=(x+1)(x-1)$.
49. $x^2-5x=3$.
50. $x^2-9x+5=0$.
51. $x^2+x=\frac{1}{4}$.
52. $x^2-7x=\frac{3}{4}$.
53. $2x(x+3)=(x+2)^2$.
54. $3x(x-1)=2(x-2)^2$.
55. $(3x+8)(x+1)=(2x+7)(x+2)$.
56. $2x^2-4x=5$.
57. $2x^2-6x=3$.
58. $2x^2-3x=2$.
59. $4x^2+8x=7$.
60. $3x^2-18x=1$.
61. $3x^2+12x=8$.
62. $2x^2+10x=11$.
63. $2x^2+2x=1$.
64. $5x^2+8x=10$.
65. $4x^2+3x=5$.
66. $6x^2-x-2=0$.
67. $6x^2+7x=20$.
68. $2x^2=4-3x$.
69. $15-8x=6x^2$.
70. $x(3x+7)=11$.
71. $3(x^2-2)=7x$.
72. $3x(x-1)=2(x+4)$.
73. $15(x-1)=4x^2-3$.
74. $3(2x^2-1)=10x$.
75. $(3x-11)(x+4)=(x-1)^2$.
76. $9x^2-30x+13=0$.
77. $x+\frac{3}{x}=7$.
78. $\frac{4x-3}{x+2}=\frac{2x+3}{x+7}$.
79. $\frac{x+2}{x+3}+\frac{x-2}{x-1}=0$.
80. $\frac{1}{x}-\frac{1}{x+1}=\frac{1}{30}$.
81. $\frac{x-1}{x-2}-\frac{x+2}{x+1}=2$.
82. $\frac{3x+4}{3x+2}-\frac{3x+2}{3x+4}=2$.
83. $\frac{4}{2x-3}-\frac{6}{3x-4}-\frac{1}{5x}=0$.

Some of the following equations have no real solution. Find them and solve the others fully.

84. $x^2+8x+59=0$.
85. $2x^2-16x+31=0$.
86. $x^2-39x+400=0$.
87. $2x^2+19x+64=0$.
88. $3x^2-23x+42=0$.

89. One side of a rectangle is 2 in. longer than the other and the area is 20 sq. in. Find the sides.

90. Find the sides of a rectangle of which the perimeter is 16 in. and the area 14 sq. in.

91. If the sum of a number and its reciprocal is 4, find the number correct to two decimal places.

92. In a certain triangle the height exceeds the base by 2 in. and the area is 20 sq. in. Find the length of the base correct to the nearest tenth of an inch.

93. Find a point P in a line AB 23 in. long, such that $AP^2 + PB^2 = 277$ sq. in.

94. Show that it is impossible to divide the line so that $AP^2 + PB^2 = 242$ sq. in.

95. The hypotenuse of a right-angled triangle is 10 in. Find the other two sides if their sum is to be 12 in.

96. Find a point P in a line AB 18 in. long such that $3AP^2 + BP^2 = 268$ sq. in.

97. When two chords AB and CD of a circle intersect at a point O inside the circle it is proved in geometry that the rectangle AO . OB is equal to the rectangle CO . OD. If AO = 3 in., BO = 4 in. and CD = 9 in., find CO and OD.

98. One train travels 5 miles per hour faster than another and in consequence takes 1 hour less to travel 150 miles. Find their speeds.

99. Find the price of tomatoes per lb. if, when the price rises 2d. per lb., one can purchase 1 lb. less for 5s.

100. Round the outside of a rectangular plot of which one side is double the length of the other runs a path $4\frac{1}{2}$ ft. wide. If the combined area of the path and plot is 770 sq. ft., find the sides of the plot.

101. The height h ft. to which a body will rise in t sec. when projected vertically upwards with a velocity of 160 ft. per sec. is given by the formula $h = 160t - 16t^2$. Determine after what times the body will be at a height of 200 ft. Explain the two solutions.

102. Find whether it is possible for a body projected vertically with a velocity of 160 ft. per sec. to rise to a height of 480 ft.

103. The ends of a string 14 in. long are attached to two points 10 in. apart. Find where the string should be plucked in order that the two portions may be at right angles.

104. The formula $r = \dfrac{c^2 + h^2}{2h}$ is used to find the height of an arc of a circle whose radius is r, $2c$ representing the length of the chord of the arc. Find the height corresponding to a chord of 16 ft. and a radius of 17 ft. Explain the two solutions.

105. If $a^2 = b^2 + c^2 - 2bcx$, find b when $a = 7$, $c = 5$, $x = \frac{1}{2}$.

106. If $R = v + \cdot 01v^2$, find v when $R = 7$.

107. A man bought a certain number of oxen for £240; if he had bought 4 more for the same sum, each ox would have cost £2 less. Find the number of oxen and the price of each.

108. Find the price of eggs per dozen if, when the price is lowered by 6d. per dozen, three dozen more can be bought for three guineas.

109. A walks a quarter of a mile per hour faster than B, and in consequence takes a quarter of an hour less to travel 15 miles. Find the rate at which each walks.

110. The sum of the two sides of a right-angled triangle is 51 in. and the hypotenuse is 3 in. longer than the greater of the two sides. Find the sides.

111. There are two railway routes connecting two towns, the distance between the towns by one route being 150 miles and by the other route 180 miles. If a train on the longer route takes half an hour less on the journey than one on the other route whose speed is 10 miles per hour less, find their respective rates and the time taken.

112. A man spends £12 in buying wine. Had the wine cost 2 - more per bottle he would have got 10 fewer bottles. Find the price per bottle.

113. ABCD is a square of side 10 in. P, Q, R, S are four points on the sides AB, BC, CD, DA respectively, such that $AP = BQ = CR = DS = x$ in. Find x if the area of the new square is three-fourths of the area of the original square.

114. Find whether it is possible to find x so that the square may be (i) one-half, (ii) two-fifths of the original square.

115. When a stream of water is projected from the nozzle of a fire-hose at a certain angle, the height y ft. of the stream of water at a horizontal distance of x ft. from the point of projection is given by $y = x\left(1 - \frac{x}{200}\right)$. If the stream of water just reaches the top of a wall 48 ft. high, find the distance of the wall from the point of projection. Explain the two solutions.

116. From a piece of wire 84 in. long, a length of $8x$ in. is cut off, and is bent into the form of a rectangle one side of which is three times the other. The remainder is bent into the form of a square. Find the dimensions of the rectangle and the square if the combined area is 217 sq. in.

117. A man rows a mile up-stream and then returns taking altogether 40 min. If his rate in still water is x miles per hour, and the speed of the current is 2 miles per hour, write down expressions for (i) the time taken going up-stream, (ii) the time taken on the return journey. Use your results to find x.

118. ABCD is a square of side x in. AB, BC, CD, DA are produced to E, F, G, H respectively, so that $BE = CF = DG = AH = 2$ in. It is easy to show that EFGH is a square. Find x, when the area of the new square is (i) twice, (ii) 3 times the area of the original square.

119. A giant weighs 3 lb. for every inch of his height and the

square of his height in feet exceeds his weight in stones by 31. Find his height and weight.

120. The area of a trapezium is given by $A = \frac{1}{2}h(a + b)$ where a and b denote the length of the parallel sides and h is the perpendicular distance between them. Find this distance when the area is 231 sq. in., if the shorter of the two parallel sides is 10 in. long, and the non-parallel sides are equal in length and each inclined to this side at an angle of 135°.

121. A man invests £7500 in £100 shares when they are below par and afterwards sells all but 60 of them for £5000 when they are as much above par as they were below par when he bought them. How many shares did he buy, and what was the price of a share?

122. A box is formed out of a square sheet of tin of side 36 in. by cutting squares of side x in. out of the corners and then turning up the edges. Find x if a rod 36 in. long will just fit diagonally into the box.

CHAPTER XIX.

LITERAL EQUATIONS AND TRANSFORMATION OF FORMULÆ.

I. HITHERTO we have discussed equations where only the unknown quantities are represented by letters. We shall now apply our methods to cases where the coefficients of terms involving the unknowns, and the constant terms, are also represented by letters, these letters representing known numbers.

As a general rule the letters at the end of the alphabet represent the unknowns, while the other letters are assumed known. Such equations are called **literal equations.**

Ex. 1.—*Solve* $p^2(x-1)=q(qx+p)$.

$$p^2(x-1)=q(qx+p)$$
$$\therefore p^2x-p^2=q^2x+pq$$
$$\therefore p^2x-q^2x=p^2+pq$$
$$\therefore x(p^2-q^2)=p^2+pq$$
$$\therefore x=\frac{p^2+pq}{p^2-q^2}$$
$$\therefore =\frac{p(p+q)}{(p+q)(p-q)}$$
$$=\frac{p}{p-q}$$

Ex. 2.—*Solve* $ab(x^2+1)=x(a^2+b^2)$.

$$ab(x^2+1)=x(a^2+b^2)$$
$$\therefore abx^2+ab=a^2x+b^2x$$
$$\therefore abx^2-a^2x-b^2x+ab=0$$
$$\therefore ax(bx-a)-b(bx-a)=0$$
$$\therefore (bx-a)(ax-b)=0$$
$$\therefore x=\frac{a}{b} \text{ or } \frac{b}{a}.$$

EXAMPLES XIX (a).

Solve:—

1. $7(x+b)=2(2x+b)$.
2. $3ax-b=ax+b$.
3. $px+q=p+qx$.
4. $ax-a^2=bx-ab$.
5. $ax+b^2=a^2-bx$.
6. $p(x-2q)=q(x+3p)$.
7. $px+2(x-q)=pq$.
8. $m(x-n)=2n(n-x)$.
9. $c(x-c)+d(x-d)=2cd$.
10. $\frac{x-a}{2}=\frac{x+b}{5}$.
11. $\frac{x-b}{b}=\frac{x-a}{a}$.
12. $\frac{x-c}{c}=\frac{x+d}{d}$.
13. $\frac{x-2b}{a}+\frac{x-2a}{b}=4$.
14. $x(x+a)-x(x+b)=(a-b)^2$.
15. $a(x-a-2b)=b(b-x)$.
16. $4x^2-a^2=(a-2x)^2$.
17. $x^2+(px+1)^2=x^2(p^2+1)$.
18. $\frac{1}{x}+\frac{1}{a}=\frac{1}{b}$.
19. $\frac{x}{a-x}=\frac{c}{b}$.
20. $\frac{x-2m-1}{m}=\frac{1-2n-x}{n}$.
21. $a(x-b+c)=a^2-x(b-c)$.
22. $\frac{x+a}{a}-\frac{x-b}{b}=\frac{a}{b}+\frac{b}{a}$.
23. $\frac{a+x}{b}=2-\frac{b-x}{a}$.
24. $\frac{a}{x-b}+\frac{b}{x-a}=\frac{a+b}{x}$.

25. The perimeter of a rectangle is p in. and one side is twice as long as the other. Find the sides.

26. A line of length a in. is divided into two parts so that one is b in. longer than the other. Find the length of each part.

27. The perimeter of a triangle is $2s$ cm. and the base is a cm. If one of the sides is three times the other, express these sides in terms of s and a.

28. Find the angles of a triangle if the vertical angle is m times each base angle.

29. Divide £p between A and B so that A's share may be $\frac{1}{n}$ of B's.

30. Divide £n among A, B, and C so that A may have twice as much as B, and B £m more than C.

31. A man walks from A to B at m miles per hour, and returns at n miles per hour. In all he takes three hours. Find the distance from A to B.

32. The distance between A and B by one route is m miles and by a shorter route is n miles. To cycle from A to B in a certain time a boy finds that his speed must be x m.p.h. by the shorter road, and 1 m.p.h. more by the longer road. Find x.

33. To one insurance company a man pays an annual premium of £x and has paid it for a years. His premium to another company is £5 more and has been paid for b years. The total sum received by the first company is £c more than that received by the second. Find x.

Solve :—

34. $3x^2 - 7ax + 2a^2 = 0$.
35. $8x^2 = 10bx + 3b^2$.
36. $a(6a - x) = 2x^2$.
37. $(x - a)^2 = b^2$.
38. $x^2 - (2a + b)x + 2ab = 0$.
39. $x^2 + mx - nx = mn$.
40. $3px^2 - 9qx = p^2x - 3pq$.
41. $ax^2 + b = x(a + b)$.
42. $ab(x^2 + 1) = x(a^2 + b^2)$.
43. $x(3a + b - 2x) = a(a + b)$.
44. $\dfrac{a + b}{a - x} = \dfrac{a + x}{a - b}$.
45. $\dfrac{5}{x - a} + \dfrac{3}{x - 5a} = \dfrac{1}{a}$.

II. Simultaneous Equations.

Ex.—*Solve the equations,* $ax + by = a^2$, $x - y = -b$

$$ax + by = a^2 \quad . \quad . \quad . \quad . \quad (1)$$
$$x - y = -b \quad . \quad . \quad . \quad . \quad (2)$$

Multiply (2) by a $\therefore ax - ay = -ab$ . . . (3)

Subtracting (3) from (1) we have $y(a + b) = a^2 + ab$

$$\therefore y = \frac{a(a + b)}{a + b}$$
$$= a.$$

From (2) $x - a = -b$

$\therefore x = a - b$

$x = a - b$
$y = a.$

EXAMPLES XIX (b).

Solve :—

1. $x + y = 5a$
   $x - y = 3a$.
2. $2x + y = 7c$
   $3x - 2y = 7c$.
3. $5x - y = 6c$
   $4x - 5y = 9c$.
4. $2ax + y = 7b$
   $ax - y = 2b$.
5. $x + 2py = 3pq$
   $2x - py = pq$.
6. $2mx + 3ny = 12mn$
   $3mx + 2ny = -7mn$.
7. $ax + by = a^2 + b^2$
   $bx - ay = 0$.
8. $x + y = 2a$
   $ax - by = a^2 + b^2$.
9. $lx - my = l^2 + lm$.
   $x - y = 2l$.
10. $px + qy = p^2 + pq$
    $qx + py = pq + q^2$.
11. $\dfrac{x + y}{2} = p + q$
    $x - 2p = y - 2q$.
12. $\dfrac{x}{a} + \dfrac{y}{b} = 2$
    $x + y = a + b$.
13. $\dfrac{2x}{a} - ay = a$
    $3x - 2a^2y = a^2$.
14. $a^2x - by = a^3$
    $bx - y = a^2$.
15. $abx - by = a^2$
    $ax + y = b$.
16. $ax + by = a^2$
    $bx - ay = b^2$.
17. $y = mx + b$
    $x = my - b$.
18. $m(x + y) = n(x - y)$
    $n(x + y) + m(x - y) = m^2 + n^2$.
19. $\dfrac{c}{x} + \dfrac{d}{y} = 1$
    $\dfrac{3c + d}{x} - \dfrac{c - d}{y} = 2$.
20. $a_1x + b_1y = a_2x + b_2y = c$.

III. Changing the Subject of a Formula.

The formula $C = 2\pi r$ gives us the circumference of a circle when the radius is known. C, being the quantity whose value we are determining, is called the **subject** of the formula. If we are given the circumference and wish to find the radius, it is convenient to have the formula written as $r = \frac{C}{2\pi}$, the subject now being r. We are said to have changed the subject from C to r.

Ex. 1.—*If* $I = \frac{PRY}{100}$, *change the subject from I to R.*

$$I = \frac{PRY}{100}$$
$$\therefore 100I = PRY$$
$$\therefore \mathbf{R = \frac{100I}{PY}}.$$

Ex. 2.—*If* $W = \frac{T}{b - T}\left(1 - \frac{a}{f}\right)$, *change the subject to T.*

$$W = \frac{T}{b - T}\left(1 - \frac{a}{f}\right).$$

Multiplying both sides by $f(b - T)$:—

$$Wf(b - T) = Tf\left(1 - \frac{a}{f}\right).$$
$$\therefore Wfb - WfT = Tf - Ta$$
$$\therefore Wfb = T(Wf + f - a)$$
$$\therefore \mathbf{T = \frac{Wfb}{Wf + f - a}}.$$

EXAMPLES XIX (c).

Change the subject in the following formulae :—

1. $A = \frac{1}{2}bh$ to h.
2. $P = 2(l + b)$ to l.
3. $p = \frac{40}{v}$ to v.
4. $l = 6 - 2\pi r$ to r.
5. $s = \frac{1}{2}(a + b + c)$ to a.
6. $P = 2r + \pi r$ to r.
7. $y = \frac{c}{x} + a$ to x.

8. $y = mx + \frac{a}{m}$ to a.

9. $F = \frac{wv^2}{gr}$ to r.

10. $s = \frac{1}{2}(u + v)t$ to u.

11. $s = \frac{a}{1 - r}$ to r.

12. $y = \frac{x + 2}{x + 3}$ to x.

13. $a = \frac{(P - Q)g}{P + Q}$ to P.

14. $V = v(1 + at)$ to a.

15. $A = P\left(1 + \frac{r}{100}\right)$ to r.

Solve each of the following equations for x:—

16. $y^2 = 4ax$.

17. $xy + a = b$.

18. $\frac{y}{x} + a = c$.

19. $a^2 = b^2 + c^2 - 2bcx$.

20. $xy + yx + fy + c = 0$.

21. $\frac{y}{x - 1} = \frac{b - c}{x}$.

22. If $v^2 = u^2 + 2as$, find s.

23. Make a the subject of the formula $\frac{F - 2}{a} = \frac{W}{g}$.

24. If $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$, prove $v = \frac{uf}{u - f}$, and find an expression for u.

25. If $t^2 = \frac{Ph}{16(P - W)}$, solve for P.

26. Make u the subject of the formula $\frac{n}{v} - \frac{1}{u} = \frac{n - 1}{r}$.

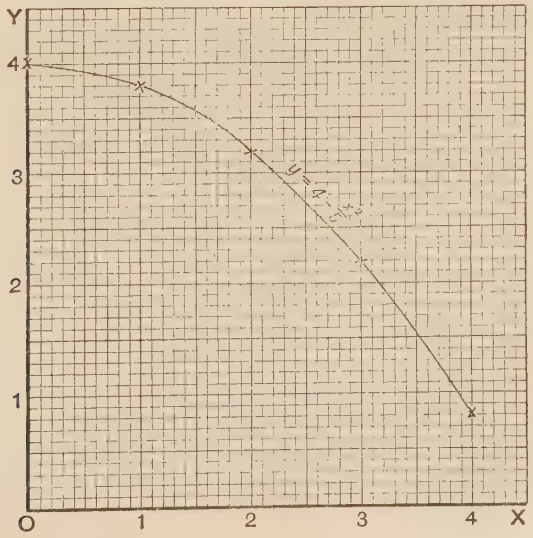
CHAPTER XX.

GRAPHS OF ALGEBRAIC FUNCTIONS.

I. Consider the following formula, $y = 4 - \frac{x^2}{5}$. Let us calculate the values of y corresponding to the values 0, 1, 2, 3, 4 of x. The results are given in the annexed table :—

x	0	1	2	3	4
y	4	3·8	3·2	2·2	·8

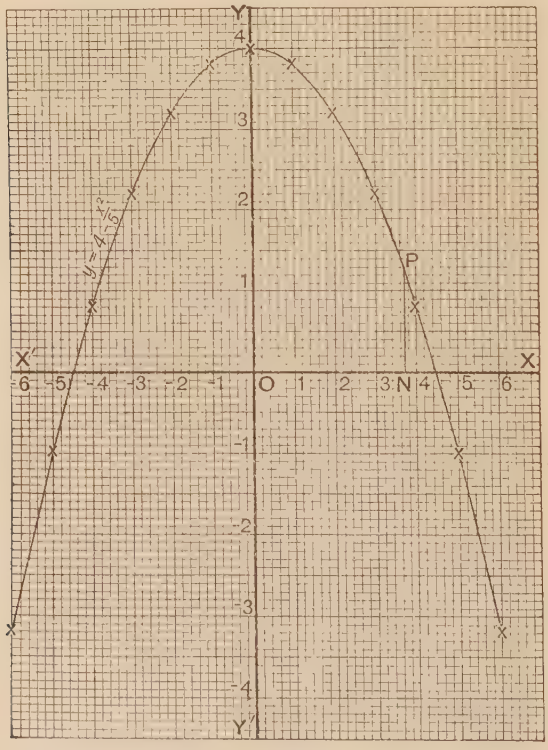
From this table a graph can be constructed as explained in Chapter X. It is shown below, the horizontal line OX being taken as the x axis, and the vertical line OY as the y axis.



Next, let us calculate the value of y when $x = -6, -5, -4, \ldots 5, 6$. The results are shown in the table below :—

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
y	$-3{\cdot}2$	-1	$\cdot 8$	$2{\cdot}2$	$3{\cdot}2$	$3{\cdot}8$	4	$3{\cdot}8$	$3{\cdot}2$	$2{\cdot}2$	$\cdot 8$	-1	$-3{\cdot}2$

When we attempt to draw a graph from this table we are faced at once with the problem of how to deal with negative numbers. We proceed as follows :—



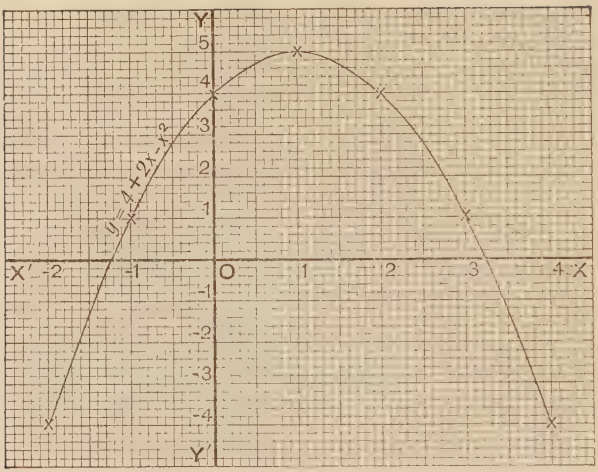
Two perpendicular lines X′X and Y′Y intersecting at O, are drawn as axes. As before, positive values of x and y are represented along OX and OY. Negative values of x are represented along OX′, and negative values of Y along OY′. Hence when we have a negative value of x, we reckon to the left instead of to the right, and when we have a negative value of y we reckon down instead of up. Now we see from the table that when $x = -5$, $y = -1$, and when $x = -4$, $y = \cdot 8$. To plot the corresponding points we reckon, (i) 5 along OX′ and 1 down, and (ii) 4 along OX′ and ·8 up. Carrying out this operation for the other pairs of values we obtain the graph shown in the diagram on page 120.

The values of x and y corresponding to a point P are called the co-ordinates of P and we write $P \equiv (x, y)$. The x co-ordinate is called the abscissa, and the y co-ordinate the ordinate. Thus in the figure ON is the abscissa and NP the ordinate of P.

Ex. 1.—*Draw the graph of* $y = 4 + 2x - x^2$ *from* $x = -2$ *to* $x = 4$. *Between what values of* x *is* y *greater than* 2? *By how much does its greatest value exceed* 2?

The corresponding values of x and y are calculated from the equation, as shown in the annexed table, and the graph is given on page 122.

x	-2	-1	0	1	2	3	4
4	4	4	4	4	4	4	4
$2x$	-4	-2	0	2	4	6	8
$-x^2$	-4	-1	0	-1	-4	-9	-16
y	-4	1	4	5	4	1	-4



From the diagram it is clear that y is greater than 2 between the values $-\cdot73$ and $2\cdot73$ of x. The maximum value of y being 5 it exceeds 2 by 3.

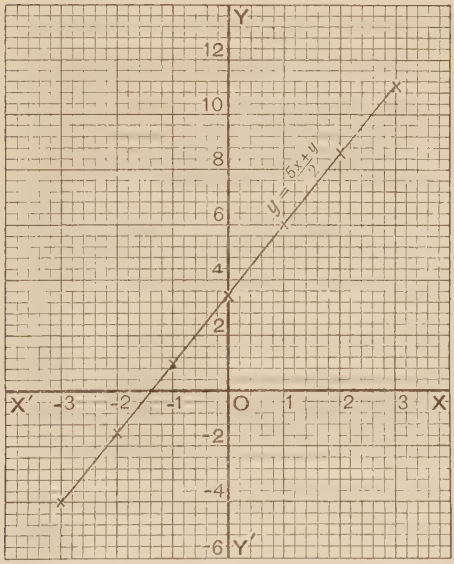
The Straight Line Law.

II. It can be proved that the **graph of any equation of the form $ax + by + c = 0$ is a straight line.** A formal proof will be given later. The examples at the end of the chapter will afford the pupil ample opportunity of testing for himself the truth of the statement. For the present we shall content ourselves with verifying this in the case $5x - 2y + 7 = 0$.

Solving the equation for y we have $y = \frac{5x + 7}{2}$, whence we can construct the following table:—

x	-3	-2	-1	0	1	2	3
y	-4	$-1\cdot5$	1	$3\cdot5$	6	$8\cdot5$	11

Plotting the corresponding points in the usual manner we find that they lie in a straight line as shown in the diagram on page 123.



To draw a straight line graph it is necessary to plot only two points on the line, as a line is completely determined when the positions of two points on it are known, but in practice it is advisable as a check to plot a third point.

The Graphical Solution of Equations.

III. The following examples will sufficiently explain the application of graphs to the solution of equations.

Ex. 1.—*Solve graphically the equation* $x^2 - 2x - 5 = 0$.

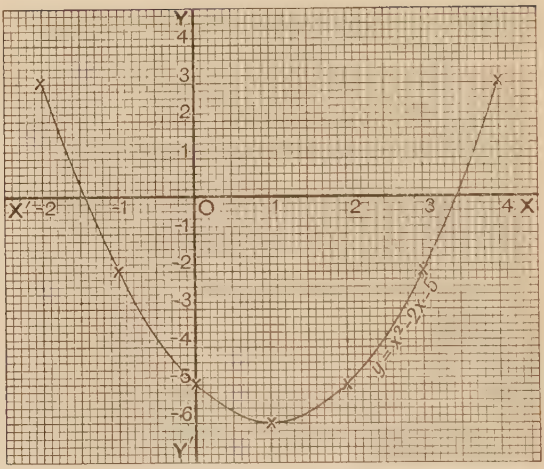
All that is necessary is to draw the graph of $y = x^2 - 2x - 5$, and consider the values of x for which $y = 0$.

From the following table :—

x	-2	-1	0	1	2	3	4
y	3	-2	-5	-6	-5	-2	3

we obtain the graph shown on page 124.

Erratum.—In the above diagram read $y = \frac{5x + 7}{2}$.



Now $y = 0$ where the graph crosses the axis of x, and the corresponding values of x are $-1{\cdot}45$ and $3{\cdot}45$. These are the roots of the equation.

Ex. 2.—*Solve from the above graph* $x^2 - 2x = 4$.

We may write this equation as $x^2 - 2x - 5 = -1$.

We must therefore find for what values of x, y is equal to -1.

From the diagram $y = -1$ when $x = -1{\cdot}24$ or $3{\cdot}24$. The roots of the equation $x^2 - 2x = 4$ are therefore $-1{\cdot}24$ and $3{\cdot}24$.

Ex. 3.—*Solve graphically* $x^3 - 11x + 10 = 0$.

This equation might be solved graphically in the manner of Example 1, by drawing the graph of $y = x^3 - 11x + 10$, but in this case we shall adopt a different plan.

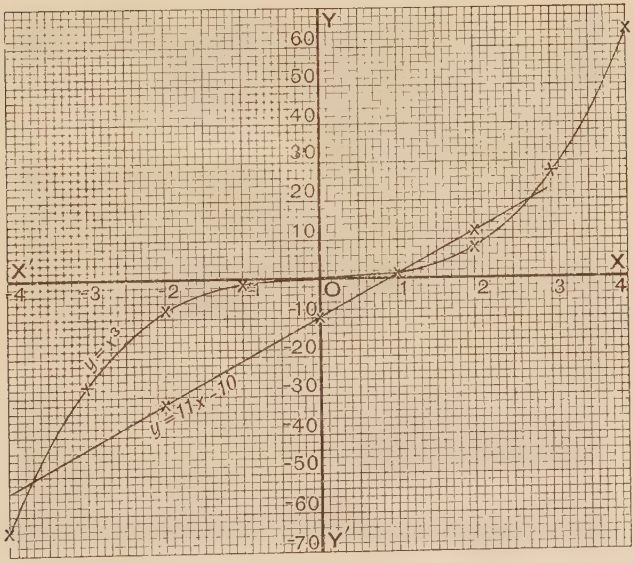
The equation may be written $x^3 = 11x - 10$. Let us then draw on the same diagram the graphs of $y = x^3$ and $y = 11x - 10$.

For $y = x^3$

x	-4	-3	-2	-1	0	1	2	3	4
y	-64	-27	-8	-1	0	1	2	27	64

For $y = 11x - 10$, the graph of which is a straight line.

x	-2	0	2
y	-32	-10	12



Evidently at the points of intersection of the two graphs the y's are equal. The abscissæ of these points therefore satisfy the equation $x^3 = 11x - 10$. Reading off from the diagram the required roots are $x = -3{\cdot}7$, and 1, and $2{\cdot}7$.

EXAMPLES XX.

1. Plot the following points on the same diagram :—

(i) (3, 2) (ii) (2, −3) (iii) (−1, 2) (iv) (−3, −3)
(v) (−1, 0) (vi) (0, −2) (vii) (1, −3) (viii) (−2, −1).

The following tables give corresponding values of x and y. For each case draw the graph showing the relation between x and y.

2.

x	-2	-1	0	1	2	3
y	-4	-2	0	2	4	6

3.

x	-2	-1	0	1	2	3
y	5	3	1	-1	-3	-5

4.

x	-3	-2	-1	0	1	2	3
y	4	0	-2	-2	0	4	10

5. Verify the fact that the graphs of the following equations are straight lines :—

(i) $y=3x$. (iii) $y+x=4$. (v) $3x+2y=6$.
(ii) $y+2x=0$. (iv) $2y-x=2$. (vi) $2x-3y=6$.

6. Draw the graph of $y=x^2$ from $x=-3$ to $x=3$. Read off from the graph the value of y when $x=2{\cdot}4$ and the value of x when $y=2$.

7. Draw the graph of $y=\frac{x^3}{10}$ from $x=-4$ to $x=4$. Read off from the graph the value of y when $x=2{\cdot}8$ and the value of x when $y=3{\cdot}2$.

8. Draw the graph of $y=x(x-3)$ from $x=-2$ to $x=4$. Find the lowest value of y and the values of x for which $y=-1$.

9. Draw the graph of $y=x(4-x)$ from $x=-1$ to $x=5$. Find the greatest value of y and the corresponding value of x. For what values of x is y equal to (i) 2, (ii) -2, (iii) -4?

10. Draw the graph of $y=x(2x-5)$ from $x=-2$ to $x=4$. Find the lowest value of y and the corresponding value of x. For what values of x is y equal to (i) 2, (ii) -2?

11. Draw the graph of $y=x+\frac{12}{x}$ from $x=1$ to $x=6$. Find the least value of y. For what values of x is y equal to 7·5?

12. Graph the expression $\frac{2x+4}{5}$ from $x=-5$ to $x=6$. From the diagram find the value of x for which $\frac{2x+4}{5}=1$.

13. In order to convert readings of temperatures from degrees

Centigrade to degrees Fahrenheit we use the relation $F. = \frac{9}{5}C. + 32$. Draw a graph from which all temperatures from $-10°$ C. to $20°$ C. can be read off in degrees Fahrenheit. Find the Fahrenheit reading corresponding to $4°$ C., and the Centigrade readings corresponding to $10°$ F. and $56°$ F.

14. A newsagent calculates that his daily profit P pence on the sale of a certain newspaper varies with the number of dozens sold, x, according to the relation $P = 3x - 10$. Exhibit this relation graphically between the values 0 and 10 of x. Find from the graph how many dozen he must sell (i) to gain 1s. 6d.; (ii) to just clear expenses; (iii) to lose 6d.

15. Draw the graphs of $y = 2 - \frac{x}{2}$ and $y = \frac{3}{2}(x - 1)$ on the same diagram, and find the co-ordinates of the point of intersection. What are the solutions of the simultaneous equations $y = 2 - x$ and $y = \frac{3}{2}(x - 1)$?

16. Solve graphically the equations $x - 2y = 2$ and $3y - 4x = 3$.

17. Draw the graph of $5 + 3x - x^2$ from $x = -2$ to $x = 5$ and hence solve $5 + 3x - x^2 = 0$.

18. Draw the graph of $y = 2x^2 + 8x - 11$ from $x = -5$ to $x = 2$. Find the lowest value of y and the corresponding value of x. Use your graph to solve the equations (i) $2x^2 + 8x = 5$, (ii) $x^2 + 4x + 2 = 0$.

19. Draw on the same diagram the graphs of $y = x^2$ and $y = x + 3$ from $x = -3$ to $x = 3$. Hence solve the equation $x^2 = x + 3$.

20. Draw the graphs of $y = \frac{4}{x}$ and $y = 6 - x$ from $x = 0{\cdot}5$ to $x = 6$. Find the x co-ordinates of the points of intersection, and write down the equation of which these are the roots. Show that it can be written in the form $x^2 - 6x + 4 = 0$.

21. Draw the graphs of $y = 6 - \frac{x^2}{2}$ and $x + 2y = 4$ on the same diagram taking 1 in. as the unit of x and $\frac{1}{2}$ in. as the unit of y. Find the abscissæ of the points of intersection and write down in a form cleared of fractions the equation of which these are the roots.

22. Taking 1 in. as the unit of x and $\frac{1}{10}$ in. as the unit of y draw the graphs of $y = x^3$ and $y = 10x - 3$ on the same diagram. Hence solve the equation $x^3 - 10x + 3 = 0$.

23. Draw the graph of $y = \frac{6}{x}$ for positive values of x from $x = 0{\cdot}5$ to $x = 3$, and for negative values of x from $x = -0{\cdot}5$ to $x = -3$. On the

same diagram draw the graph of $x - y = 2$. Hence solve the equation $x^2 - 2x = 6$. [Note that the graph of $y = \frac{6}{x}$ consists of two distinct branches.]

24. Taking half an inch as unit draw the graphs of (i) $y = \frac{8}{x^2}$ from $x = -4$ to $x = -1$, and from $x = 1$ to $x = 6$, (ii) $x + 2y = 6$ from $x = -4$ to $x = 6$. Write down the co-ordinates of the points of intersection, and show that the x co-ordinates of these points are the roots of the equations $x^3 + 16 = 6x^2$.

25. Solve graphically the equation $x^3 - 4x^2 + 4 = 0$ by writing the equation in the form $4x - x^2 = \frac{4}{x}$ and drawing two separate graphs. Take values of x ranging from $x = -2$ to $x = 5$.

CHAPTER XXI.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

I. When one Equation is Linear.

Ex. 1.—*Solve the equations :—*

$$x^2 - xy + 2y^2 = 8 \quad . \quad . \quad . \quad (1)$$
$$3x - 2y = 7 \quad . \quad . \quad . \quad (2)$$

Solve equation (2) for x

$$\therefore x = \frac{2y + 7}{3} \quad . \quad . \quad . \quad . \quad (3)$$

Substitute for x in (1)

$$\therefore \frac{(2y + 7)^2}{9} - \frac{y(2y + 7)}{3} + 2y^2 = 8$$
$$\therefore (2y + 7)^2 - 3y(2y + 7) + 18y^2 = 72$$
$$\therefore 4y^2 + 28y + 49 - 6y^2 - 21y + 18y^2 = 72$$
$$\therefore 16y^2 + 7y - 23 = 0$$
$$\therefore (y - 1)(16y + 23) = 0$$

$$\therefore y = 1 \text{ or } -\frac{23}{16}.$$

Substituting in (3) we have

when $y = 1$, $x = 3$

„ $y = -\frac{23}{16}$, $x = \frac{11}{8}$

x	3	$\frac{11}{8}$
y	1	$-\frac{23}{16}$

We might have solved equation (2) for y instead of for x, and so have eliminated y from equation (1).

Sometimes it will be found that one of the variables is more easily eliminated than the other, in which case equation (2) should be solved for that variable.

While the above method is applicable to all cases in which one of the equations is linear, there are some types which can be more easily solved by special methods.

Ex. 2.—*Solve the equations :—*

$$x^2 + y^2 = 265 \quad . \quad . \quad . \quad . \quad (1)$$
$$x + y = 23 \quad . \quad . \quad . \quad . \quad (2)$$

Square equation (2) $\therefore x^2 + 2xy + y^2 = 529.$
Subtract equation (1) from this result.

$$\therefore 2xy = 264 \quad . \quad . \quad . \quad . \quad (3)$$

Subtract (3) from (1).

$$\therefore x^2 - 2xy + y^2 = 1$$
$$\therefore (x - y)^2 = 1$$
$$\therefore x - y = 1 \text{ or } -1.$$

$\therefore$ either $x + y = 23$	or $x + y = 23$
$x - y = 1$	$x - y = -1$
whence $x = 12$	whence $x = 11$
and $y = 11$	$y = 12$

x	12	11
y	11	12

Ex. 3.—*Solve the equations :—*

$$x^3 - y^3 = 387 \quad . \quad . \quad . \quad . \quad (1)$$
$$x - y = 3 \quad . \quad . \quad . \quad . \quad (2)$$

Since $x^3 - y^3 = 387$

$$\therefore (x - y)(x^2 + xy + y^2) = 387.$$

But $x - y = 3$ by equation (2).

$$\therefore x^2 + xy + y^2 = 129 \quad . \quad . \quad . \quad (3)$$

We may now proceed as in Example (1), using equations (2) and (3), or as follows:—

Square (2) $\therefore x^2 - 2xy + y^2 = 9.$

Subtract this result from equation (3)

$$\therefore 3xy = 120$$
$$\therefore xy = 40.$$

Add this result to equation (3)

$$\therefore x^2 + 2xy + y^2 = 169$$
$$\therefore (x + y)^2 = 169$$
$$\therefore x + y = 13 \text{ or } -13.$$

$\therefore$ either $x + y = 13$	or $x + y = -13$
$x - y = 3$	$x - y = 3$
whence $x = 8$	whence $x = -5$
and $y = 5$	and $y = -8$

x	8	-5
y	5	-8

EXAMPLES XXI (a).

Solve:—

1. $x - 2y = 3$
   $xy = 2.$
2. $y^2 - x = 2$
   $x = y.$
3. $x + 2y = 18$
   $xy = 40.$
4. $x^2 + y^2 = 5$
   $x + y = 3.$
5. $x^2 + y^2 = 25$
   $x + 3y = 5.$
6. $x^2 - y^2 = 3$
   $x + 2y = 4.$
7. $2x = 5y - 2$
   $x = y^2.$
8. $2x^2 - xy = 20$
   $x + y = 7$
9. $y = 2x + 1$
   $x^2 + y^2 = 10.$
10. $x + y = 5$
    $2x^2 - 3y^2 = 6.$
11. $x^2 - 2y^2 = 7$
    $2x + y = 7.$
12. $x^2 + xy + y^2 = 7$
    $x - y = 5.$
13. $3x^2 - 4xy = 3$
    $2y - x = 1.$
14. $2y + x = 2xy$
    $3x - y = 5.$
15. $4x - y = 4$
    $15x^2 - y^2 = 44.$
16. $y^2 = 2x + 3$
    $x + y + 2 = 0.$
17. $x^2 + y^2 - y = 10$
    $x + y = 5.$
18. $x^2 + xy + 1 = 0$
    $x - y = 3.$

19. $3x^2 - xy - y^2 = 12$
$2x - y = 6.$

20. $x^2 + 5xy = y + 13$
$4x - y = 1$

21. $x + 3y = 8$
$x^2 - 3xy + 9y^2 = 19.$

22. $x^2 + y^2 - 5x - 3y + 6 = 0.$
$y = 2x - 1.$

23. $xy = 0$
$6x - 5y = 30.$

24. $x^2 + 2xy = 3$
$2x + 3y = 5.$

25. $x^2 - y^2 = 7$
$3y - 2x = 1.$

26. $4x^2 - 5xy - y^2 = 10$
$5x + 2y = 1.$

27. $x^2 + 2xy = 20$
$2y - 3x = 2.$

28. $x^2 + y^2 = 170$
$x + y = 18.$

29. $xy = 299$
$x - y = 10.$

30. $x^2 + y^2 = 290$
$x - y = 24.$

31. $x^3 + y^3 = 152$
$x + y = 8.$

32. $x^3 - y^3 = 63$
$x - y = 3.$

33. $x^3 - 8y^3 = 152$
$x - 2y = 2.$

34. $x^3 + 27y^3 = 189$
$x + 3y = 3$

35. $5x^2 - 10xy + 3y^2 + 3y + 10 = 0$
$5x = 2y + 2.$

36. $2x^2 + 5xy - 4x + 5y = 7$
$4x + 5y = 1.$

37. $x^2 + y^2 - 4x + y - 27 = 0$
$3x = 4y - 17.$

38. $\frac{6}{x} - \frac{3}{y} = \frac{1}{2}$
$3y - 2x = 1.$

39. $x^2 + y^2 = 4xy - 26$
$3y - 2x = 1.$

40. $x + y + 2 = 0$
$2x^2 + 3y^2 - 14x + 2y = 65.$

II. We shall now show how to solve a pair of simultaneous equations, in which all the terms involving the unknowns x and y, are of the second degree.

Ex. 1.—*Solve the equations* :—

$$2x^2 - 3xy + 2y^2 = 14 \quad . \quad . \quad . \quad (1)$$
$$x^2 - xy + y^2 = 13 \quad . \quad . \quad . \quad (2)$$

Since $\frac{2x^2 - 3xy + 2y^2}{x^2 - xy + y^2} = \frac{14}{13}$,

$\therefore 26x^2 - 39xy + 26y^2 = 14x^2 - 14xy + 14y^2$

$\therefore 12x^2 - 25xy + 12y^2 = 0$

$\therefore (4x - 3y)(3x - 4y) = 0$

$\therefore x = \frac{3}{4}y$ or $\frac{4}{3}y.$

(i) If $x = \frac{3}{4}y$

then substitution in equation (2) gives

$$\frac{9}{16}y^2 - \frac{3}{4}y^2 + y^2 = 13$$

$$\therefore \frac{13}{16}y^2 = 13$$

$$\therefore y^2 = 16$$

$$\therefore y = 4 \text{ or } -4.$$

But $x = \frac{3}{4}y$

$$\therefore x = 3 \text{ or } -3.$$

(ii) If $x = \frac{4}{3}y$

then substitution in equation (2) gives

$$\frac{16}{9}y^2 - \frac{4}{3}y^2 + y^2 = 13$$

$$\therefore \frac{13}{9}y^2 = 13$$

$$\therefore y^2 = 9$$

$$\therefore y = 3 \text{ or } -3.$$

But $x = \frac{4}{3}y$

$$\therefore x = 4 \text{ or } -4.$$

x	3	-3	4	-4
y	4	-4	3	-3

EXAMPLES XXI (b).

Solve :—

1. $x^2 + y^2 = 10$
   $xy = 3$.
2. $x^2 + y^2 = 17$
   $xy = 4$.
3. $3x^2 + 5xy + 3y^2 = 5$
   $x^2 + y^2 = 5$.
4. $x^2 + 3y^2 = 12$
   $xy + y^2 = 4$.
5. $8xy - 3y^2 = 21$
   $x^2 - xy + y^2 = 7$.
6. $xy + 3y^2 = 4$
   $x^2 + 3xy + 4y^2 = 8$.
7. $3x^2 - 5y^2 = 22$
   $2x^2 + 3y^2 = 21$.
8. $x^2 + xy = \frac{3}{2}$
   $xy + y^2 = \frac{3}{4}$.
9. $x^2 - y^2 = 3$
   $xy + y^2 = 3$.
10. $3x^2 + 2xy = 39$
    $3xy + 2y^2 = 26$.
11. $x^2 - y^2 = 24$
    $xy - y^2 = 10$.
12. $x^2 - 2xy = 6$
    $x^2 - 4y^2 = 3$.
13. $36x^2 - 9y^2 = -32$
    $6xy + 3y^2 = 8$.
14. $2xy - 5y^2 = 3$
    $2x^2 - 5xy = 12$.
15. $9x^2 - 4y^2 = 50$
    $3xy + 2y^2 = 12\frac{1}{2}$.
16. $2x^2 - 3xy + 2y^2 = 14$
    $x^2 - xy + y^2 = 13$.
17. $3x^2 - y^2 = 3$
    $3xy + 2y^2 = 36$.
18. $2y^2 - xy = 15$
    $3y^2 - x^2 = 26$.
19. $2xy + 5y^2 = 21$
    $2y^2 - 3xy = 54$.
20. $x^2 - 5xy + 4y^2 = 10$
    $xy - 5y^2 = 1$.
21. $x^2 + 3y^2 = 7$
    $2x^2 - 3xy - 7y^2 = 7$.

22. $3xy - 5y^2 = 10$
$x^2 - xy - y^2 = 11.$

23. $5x^2 - 5xy + y^2 = 44$
$3x^2 - 3xy + y^2 = 28.$

24. $3x^2 + 12xy + 7y^2 = 7$
$3xy + 2y^2 = 2.$

25. $3x^2 + 5xy = 22$
$11xy - 3y^2 = 19.$

26. $x^2 + 5xy - 2y^2 = 34$
$2y^2 - 3xy = 5.$

27. $x^2 + y^2 = 5$
$x^2 - 2xy - y^2 = 7.$

28. $x^2 - xy = 24$
$xy + y^2 = 16.$

29. $\frac{1}{5}(x^2 + y^2) = xy - y^2 = 1.$

30. $x^2 - 3xy + 2y^2 = 6$
$x^2 - xy + y^2 = 21.$

III. Miscellaneous Equations.

Ex. 1.—*Solve the equations :—*

$$x^3 - y^3 = 351 \quad . \quad . \quad . \quad (1)$$
$$x^2 + xy + y^2 = 39 \quad . \quad . \quad . \quad (2)$$

From equation (1) $(x - y)(x^2 + xy + y^2) = 351.$
$\therefore 39(x - y) = 351$ (by eqn. 2)
$\therefore x - y = 9 \quad . \quad . \quad . \quad . \quad . \quad (3)$

On solving equations (2) and (3) in the usual manner we find that the solutions are

x	7	2
y	−2	−7

Ex. 2.—*Solve the equations :—*

$$x^4 + x^2y^2 + y^4 = 481 \quad . \quad . \quad . \quad (1)$$
$$x^2 - xy + y^2 = 13 \quad . \quad . \quad . \quad (2)$$

Since $x^4 + x^2y^2 + y^4 = (x^2 - xy + y^2)(x^2 + xy + y^2)$, (see Chap. XIV., Examples (*c*), No. 87), therefore from equation (1) we have

$$13(x^2 + xy + y^2) = 481$$
and $$\therefore x^2 + xy + y^2 = 37 \quad . \quad . \quad . \quad (3)$$

On solving (2) and (3) we find that the solutions are

x	4	−4	3	−3
y	3	−3	4	−4

Ex. 3.—*Two numbers are such that the sum of the first and the reciprocal of the second is* 2, *while the reciprocal of the first diminished by the second is* $\frac{3}{4}$. *Find the numbers.*

Let the numbers be x and y.

$$\therefore x + \frac{1}{y} = 2 \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{and } \frac{1}{x} - y = \frac{3}{4} \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{From (1)} \quad xy + 1 = 2y \quad . \quad . \quad . \quad (3)$$

$$\text{,, (2)} \quad 1 - xy = \frac{3}{4}x \quad . \quad . \quad . \quad (4)$$

$$\text{Add } \therefore 2 = \frac{3}{4}x + 2y$$

$$\therefore 8 = 3x + 8y \quad . \quad . \quad . \quad (5)$$

On solving equations (3) and (5) we find that

x	$\frac{2}{3}$	4
y	$\frac{3}{4}$	$-\frac{1}{2}$

The numbers are $\frac{2}{3}$ and $\frac{3}{4}$.

Solutions other than positive ones are usually disregarded, as problems such as the one now in question have an arithmetical interest only.

EXAMPLES XXI (c).

Solve :—

1. $2x + \frac{1}{y} = 3$
   $xy = \frac{1}{2}$.

2. $y = 2x - \frac{4}{x}$
   $2y - 3x = 2$.

3. $y^2 = 8x$
   $3y + 4x = 20$.

4. $x + \frac{4}{y} = y - \frac{2}{x} = 2$.

5. $\frac{1}{x} - \frac{1}{y} = \frac{3}{40}$
   $x - y + 3 = 0$.

6. $(x - 3)(y + 1) = 4$
$xy = 12.$

7. $(x - 2)(y + 3) = 10$
$xy = -7.$

8. $(x + 1)(y + 1) = 8$
$(x - 1)(y + 2) = 6.$

9. $(5 - x)(1 - y) = 6$
$(4 - x)(3 + y) = 1.$

10. $x^3 + y^3 = 407$
$x^2 - xy + y^2 = 37.$

11. $x^3 - 8y^3 = 72$
$x^2 + 2xy + 4y^2 = 12.$

12. $x^4 + x^2y^2 + y^4 = 133$
$x^2 - xy + y^2 = 7.$

13. $x^3 + 27y^3 = 37$
$x^2 - 3xy + 9y^2 = 37.$

14. $x^4 + x^2y^2 + y^4 = 243$
$x^2 + xy + y^2 = 9$

15. $\dfrac{1}{x-2} + \dfrac{1}{y-1} = 2$
$3x - 4y = 1.$

16. $x^2 - 5xy + 4y^2 = 0$
$(x + y)^2 - (x - y)^2 = 16$

17. $x^4 + x^2y^2 + y^4 = 481$
$x^2 + xy + y^2 = 13$

18. $xy(x + y) = 330$
$xy(x - y) = 30$

19. $x^3 + y^3 = 152$
$xy(x + y) = 120.$

20. $x^2y^2 + 12 = 7xy$
$x + y = 4.$

21. The difference of the squares of two numbers is 11. Twice the greater diminished by the smaller is 7. Find the numbers.

22. Three times the product of two numbers diminished by the square of the greater is 5. The greater diminished by twice the smaller is 1. Find the numbers.

23. The area of a rectangle is 30 sq. in. If the length is increased by 1 in., and the breadth decreased by 2 in., the area is 17 sq. in. Find the original length and breadth.

24. Two numbers are such that the square of the first diminished by four times the square of the second is 20, while the sum of the first and twice the second is 10. Find the numbers.

25. Two numbers are such that the first is equal to one-fourth of the square of the second, and also equal to its own square diminished by twice the square of the second. Find the numbers.

26. Two numbers are such that the first together with the reciprocal of the second is $1\frac{1}{6}$, while the second together with the reciprocal of the first is $4\frac{1}{2}$. Find the numbers.

27. The first of two numbers exceeds the reciprocal of the second by one-fourth, and the second exceeds the reciprocal of the first by two-thirds. Find the numbers.

28. The product of two numbers is one-third, and the difference of their reciprocals is one-half. Find the numbers.

29. One side of a right-angled triangle is 2 in., and the area of the rectangle contained by the hypotenuse and the remaining side is $3\frac{3}{4}$ sq. in. Find the sides.

30. A plank is cut into four equal lengths, and the parts are placed with their longer edges in contact, so as to form the lid of a box of section 6 sq. ft., and girth 10 ft. Find the dimensions of the plank.

31. The product of two numbers is 5 and the difference of their squares is $2\frac{1}{4}$. Find the numbers.

32. The sum of two numbers is two-thirds of their product. If the square of the smaller is two-thirds of the greater. Find the numbers.

33. A ladder 20 ft. long rests with one end on the ground, and the other against a vertical wall. If the top end slipped down 4 ft., the lower end would slip 4 ft. along the ground. How far is the top of the ladder from the ground?

TEST PAPERS.

Test Paper 1.

1. If $y = 3x - 1$, find the value of y when (i) $x = 7$; (ii) $x = 4$; (iii) $x = 3\frac{1}{3}$.

2. If $3t + 7 = 13$, find the value of t. Deduce the value $\frac{t}{2} + 5$.

3. Add the expressions $a + 2b$; $5a + 6b$; and $7a + 4b$.

4. Make up the following bill giving the value in pence :—

x lb. of tea @ 4/- per lb.
x ,, ,, jam ,, 1/- ,,
x oz. ,, cinnamon @ 2d. per oz.
x eggs @ 5d. each.

5. 1 cwt. of sugar costs 80/-. What is the value of (i) x cwt.; (ii) y stones?

Test Paper 2.

1. If $x = 2$, $y = 3$, and $z = 4$, find the value of (i) $5x - 4y + 6z$; (ii) $x^2 + z^2$; (iii) $y^2z^2 + z^2x^2 - x^2y^2$.

2. A motor car uses y gallons of petrol per mile. How many gallons will it use in (i) 30 miles; (ii) 50 miles?

3. Sum the following expressions: $5p - 3q$; $7p - 4q$; $9p - 2q$; $6p - 8q$.

4. At the beginning of a year a boy weighed 8 stones. At the end of the year he was x lb. heavier. Express his weight at the end of the year (i) in lb.; (ii) in stones.

5. A man makes m/- per hr., his wife makes w/- per hr., and their boy makes b/- per hr. One day the man works 8 hr., the woman works 5 hr., and the boy works 2 hr. Find the total of their earnings for the day.

Test Paper 3.

1. If $a = 0$, $b = 2$, $c = 3$, and $d = 4$, find the value of :—

(i) $2a + 4b - 6c + 8d$; (ii) $abc + abcd$; (iii) $\frac{1}{4}acd + \frac{1}{6}bcd - 2\frac{b^2}{d}$.

2. Ground is rented at £x per sq. pole. What will be the rent of 1 acre?

3. Sum the following expressions :—

$2a - 3b + 4c$; $5a - 7b + 2c$; $6a - 8b + 3c$; $a - b + c$.

4. From $5x + 7y + 8z$ take $2x + 3y + z$.

5. How many degrees East does a ship voyage in sailing from longitude $x°$ W. to longitude $y°$ E.?

Test Paper 4.

1. (i) Find the value of $1 + \frac{24}{x}$, when $x = 2$, 4, and 8; (ii) If $t = 5$, prove that $7t - 6 = 5t + 4$.

2. Add up the following expressions :—

$x^3 - x^2 + x - 6$; $3x^3 + x - 4$; $-6x^2 - 8$.

3. From $9a^3 - 8a^2 - 7a + 5$ take $2a^3 - 5a^2 + 3a + 1$.

4. Simplify $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}$.

5. A man's salary increases by £x annually. If it is now £275 what will it be in 6 years? What was it 2 years ago?

Test Paper 5.

1. If $x = 3$ and $y = 5$, show that $x^2 + y^2 = 2xy + 4$.

2. Simplify the following expressions :—

(i) $3t - 7t - 9t + 5t - 2t + 6t$.
(ii) $2a - 5b + 7a - 3b - 9a + 11b$.

3. If $\frac{20}{n+3} = 4$, find the value of n.

4. From the sum of $7a + 2b$ and $-a + 8b$ take $9a - 2b$.

5. A motor car consumes 1 gal. of petrol per 20 miles. If it started with x gal. of petrol in the tank, how much was left after it had done 55 miles?

Test Paper 6.

1. If $x = 3$ and $y = -2$, find the value of :—

(i) $x^2 + xy$; (ii) $5x^2 - 4xy$; (iii) $4x^2 - 7y^2$; (iv) $x^2 - xy + y^2$.

2. When $a = 5$, $b = -4$ and $c = 2$, find which of the following expressions is the greatest: $a^3 + b^3$; $a^2 + b^2 + c^2$; $-2abc$.

3. Simplify: $-9p^2 + 3q^2 - 4r^2 + 2p^2 + 7r^2 - 8p^2 - 6q^2 - 5r^2 + q^2$.

4. Simplify the following expressions :—

(i) $(a + 4b - 3c) - (5a + 2b - 7c)$.
(ii) $6(x - 2y + 3z) - 2(3x + 5y - 6z)$.

5. x lb. of coffee are adulterated with x oz. of chicory. Find the weight of the mixture (i) in oz.; (ii) in lb.

Test Paper 7.

1. If $a = -6$, $b = 2$, and $c = -5$, simplify $(4a - 3b - 6c)(a + 8b + 7c)$ and prove that $a^2 - 2a - 48 = 0$.
2. From $-6w^2 + 5w - 3$ take $4w^2 - 2w + 6$.
3. Multiply $x^2 - 8x - 3$ by $2x - 2$.
4. Divide $3y^3 - 5y^2 + 6y - 2$ by $y - 6$.
5. An employer employs m men and w women. The men he pays 9/- a day each, and the women 5/- a day each. How much will he pay his employees for a week of six days?

Test Paper 8.

1. Simplify the following expressions:—

$$\text{(i) } \frac{z}{4} + \frac{3z}{4}; \qquad \text{(ii) } \frac{x^9}{x^6}; \qquad \text{(iii) } \frac{21x^5}{-7x^2};$$

$$\text{(iv) } (5a - 4b) - 6(7a + 2b).$$

2. From the sum of $5x^3 - 7x^2 + 2x - 3$ and $9x^2 - 8x - 6$ take $7x^3 - 9x + 4$.
3. Multiply $x^3 - 6x^2 - 4x + 2$ by $2x - 3$.
4. Divide $3m^3 - 4m^2 + 5m - 8$ by $m - 2$.
5. From the basement to the landing there are 9 steps on a stair, each step being n in. deep. How many feet is the landing above the basement?

Test Paper 9.

1. If $m^3 + 2m^2 + 3m - 6 = 0$, prove that m cannot be -2.
2. Simplify the following expressions:—

$$\frac{5}{6}m - \frac{2}{3}m; \qquad \frac{x^7}{x^2} \times \frac{x^5}{x^3};$$

$$\text{and } -7(s^2 - t^2) + 3(4s^2 - 5t^2) - 2(-2s^2 + 3t^2).$$

3. A rectangular patch of ground is $(x + 5)$ chains long and $(x + 2)$ chains wide. Find the perimeter and area of the patch.
4. Divide $2x^3 - 11x^2 + 11x + 10$ by $2x - 5$.
5. A man takes a railway journey with his wife and two children. For himself and his wife he pays x/- each, while his children travel "half fare." If he started off with 40/- in his pocket, how much had he after paying for the railway tickets?

Test Paper 10.

1. Find the value of $(v - u)^2 + (3v - 2u)^2 - (5 - u + v)^2$, when $v = 8$ and $u = -2$.

2. Simplify (i) $6x^2 \times -2x^5 \times -3x^7$; (ii) $\frac{10b^9c^7}{4bc^3} \times \frac{12b^6c^2}{5b^3c^2}$.

3. Add the expressions $3p - 2q + 4r$; $-5p + 6q - r$; and $p - q + 7r$. Take the result from $2p - q - 5r$.

4. Multiply $x^4 + x^2 - 2x - 4$ by $x + 8$.

5. The height of a triangle is x in. and its base is $(8x - 2)$ in. Find its area.

If the remaining sides are $(6x + 5)$ in. and $(7x - 4)$ in. long respectively, find the perimeter of the triangle.

Test Paper 11.

1. Show that $\frac{2 - t}{8 + t} = \frac{5t + 2}{t - 1}$, when $t = -6$ or $-\frac{1}{2}$.

2. Simplify (i) $5(x^2 - 2x - 4) - 6(2x^2 - 5x + 2)$;

(ii) $6\left(\frac{1}{3}x - \frac{1}{2}y\right) - 12\left(\frac{5}{6}x + \frac{3}{4}y\right)$.

3. Multiply $x^3 - 2x^2 + 5x - 3$ by $x^2 + 2x - 4$.

4. Divide $x^3 - 6x^2 + 5x - 12$ by $x - 4$.

5. Solve (i) $5x - 9 = 12$; (ii) $7t + 4 = 9t - 6$.

Test Paper 12.

1. If $y = 3x + 7$, complete the following table :—

x	0	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{3}$	4	$4\frac{1}{6}$	5
y										

2. Reduce c ch. to feet; y yd. to inches; s stones to ounces; £p + s shillings + d pence to pence.

3. (i) Simplify $\frac{1}{2}(2a - 4b) - \frac{1}{3}(9a + 6b) + \frac{1}{5}(25a - 15b)$.

(ii) Multiply $2x^3 - x^2 + 4x - 3$ by $2x - 7$.

4. Solve (i) $1{\cdot}2m = 24$; (ii) $\frac{36}{x} = 9$; (iii) $\frac{42}{w} - 10 = 4$.

5. Seven is added to three times a certain number x. Express the result algebraically. If the result is 19, find x.

Test Paper 13.

1. If $x = 3y + 20$, find the value of x when $y = -6$.
If $3v = 5 + 7f$, find the value of v when $f = -2$.

2. Subtract $3x^2 - 5x - 6$ from $2x^2 + 9x - 11$.

3. Prove that $y^3 + 5y^2 - 22y + 7 = (y - 3)(y^2 + 8y + 2) + 13$.

4. Solve (i) $2x + 4{\cdot}6 = 8{\cdot}4$; (ii) $2y = 3x + 8$ when $y = -5$; (iii) $6w - 7 = 4w - 16$.

5. In $\triangle ABC$, $\hat{A} = (3x - 6)$ deg.; $\hat{B} = (68 - 2x)$ deg. and $\hat{C} = (7x + 14)$ deg. Express the sum of these angles in terms of x and then find the size of each.

Test Paper 14.

1. If $y = \frac{x}{2} - 6$, complete the following table :—

x	-5	-4	-1	0	3	5	7
y							

2. Simplify $\frac{2}{3}(9x - 6y) - \frac{3}{4}(4x + 12y)$.

3. Multiply $-7x^3 - 2x^2 + 3x - 8$ by $-2x + 4$.

4. Solve the following equations :—

(i) $10z = \frac{5}{2}$. (ii) $\frac{t}{21} = \frac{1{\cdot}6}{1{\cdot}4}$.

(iii) $7(x - c) = 4(3x + 2c)$ given that $c = 2$.

5. One side of a rectangle is x in., and an adjacent side is 5 in. longer. What is the perimeter? If the perimeter is 70 in., find the sides of the rectangle.

Test Paper 15.

1. If $y = 3x + 4{\cdot}4$, fill in the following table giving the values of y corresponding to those of x.

x	-2	$-1\frac{1}{2}$	$-\frac{1}{3}$	0	$3\frac{2}{3}$	$4\frac{1}{6}$
y						

2. (i) Add $5x - 4y + 11z$, $8x + 3y - z$, $7x + 9y$, $-4x - 2y + 5z$.
(ii) From $3a^2 - 6a - 4$ take $7a^3 + 2a^2 - 9a + 6$.

3. Multiply $x^2 + x + 1$ by $x^2 - x + 1$ and divide the result by $x - 4$.

4. Solve (i) $-\frac{3}{7}x = \frac{9}{14}$. (ii) $y - 9 = 5y - 3$.

(iii) $8x - 3 + 7x - 9 = 16x + 8 - 6x$.

5. A wire of length 15 in. was cut into two parts, such that one was $\frac{3}{7}$ of the other. Find the length of each part.

Test Paper 16.

1. (i) How many stones are there in c cwt. ?

(ii) ,, ,, days ,, ,, h hr. ?

(iii) From a Relief Fund each man gets 1/- per day, each woman 8d., and each child 3d. Find in shillings how much in all m men, w women, and c children, receive in a week.

2. Simplify $4(2x^2 - 3x + 6) - 5(3x^2 - 8x - 2) + 6(x^2 - 5x - 7)$.

3. Simplify $(x + 1)(x^3 - 2x^2 + x - 3) + 2x(x - 4)$.

4. Solve $2(3x - 5) - 3(x - 4) = 8 - x$.

5. A number x is chosen. Multiply it by 3 and take 5 from the product. The result is the same as that obtained by adding 9 to the number. What is the number ?

Test Paper 17.

1. Evaluate $(x+2y)^2-(2y+3z)^2+(3z+x)^2$ when $x=1$, $y=\frac{1}{2}$, and $z=\frac{1}{3}$.

2. (i) A man weighs x stones and a boy weighs half as much. What will they weigh together ?

(ii) Find the value of x lb. of tea @ 4/- per lb. ; y lb. of coffee @ 2/- per lb. ; and z lb. of cocoa @ 3/- per lb.

3. Simplify $[(x^3 - x^2 - 5x + 6) \div (x - 2)] - 3(x - 6)$.

4. (i) If $y = 14x - 5z$ find x when $y = 2$ and $z = -6$.

(ii) Solve $2(t-3)+5(t+1)=3(2t-1)+2(t-2)$.

5. On receiving 6/- a boy has 3 times what he had at first. How much did he have originally ?

Test Paper 18.

1. If $a=0$, $b=-2$, $c=3$, $d=-4$, find the value of (i) $\sqrt{a^2+2bd}$; (ii) $\sqrt{a^2+c^2+d^2}$.

2. (i) If n apples at xd. each cost C/-, find the formula connecting C, n, and x.

(ii) In $\triangle$ ABC $\hat{A} = (x+4)$ deg. and $\hat{B} = (7 - 5x)$ deg. Find the size of C.

3. Prove that $(x^2+x-2)(x-4)=(x^4-15x^2-10x+24) \div (x+3)$.

4. (i) If $2T = h(a + b)$ find a when $T = 60$, $h = 6$, and $b = 13$.

(ii) Solve $2(m-3)+2(4m+5)=3(5m+4)-6(m+1)$.

5. If the sides of an equilateral triangle were each shortened by 2 in., the perimeter would be half of what it would be if they were lengthened 3 in. Find the sides of the triangle.

Test Paper 19.

1. A household uses c cwt. of coal per week. How many tons is this per year? What is the cost at £3 per ton?

2. Prove that the sum of $-3p+2q-4r$, $-6q+7r$, and $-4p+9r$ is equal to the excess of $5p-3q+r$ over $12p+q-11r$.

3. Expand $(x-1)(x+2)(x-3)(x+4)$.

4. (i) If $5a-2=2a-20$ and $6b-5=4b+3$, find the value of a^2-ab+b^2.

(ii) Solve $2(y-3)+4(2y+7)=2(5y+9)-5(y+1)$.

5. In $\triangle$ ABC $\hat{A}=2\hat{B}$, and $\hat{C}$ is 20° greater than $\hat{B}$. Find the angles of the triangle.

Test Paper 20.

1. If $y=5x-8$ and $z=3y+7$ find the value of z when $x=3$, 7, -2 and -8 respectively, and tabulate your answers.

2. Prove that—

$9(3a-2b+4c)-5(4a+b-6c)=8(5a-4b+9c)-3(11a-3b+2c)$.

3. Divide $3c^4+5c^3-5c^2-5c+7$ by $3c-1$.

4. Solve $2\sqrt{x}=18$ and $9(3t-2)-4(5t+6)=8(2t-7)-6(4t-5)$.

5. The length of a rectangle is 9 in. and its breadth is x in. If the length be increased 3 in. and the breadth decreased 1 in., its area will be unaltered. Find the width.

Test Paper 21.

1. The equal sides of an isosceles triangle are each b in. long, and the base is a in. long. The height, h in., is given by the formula $h=\sqrt{b^2-\frac{a^2}{4}}$. Find h when $a=6$ and $b=5$.

2. A boy spends one-half of his money in one shop, and one-third of it in another. He has x/- left. What had he at first?

3. Solve (i) $\cdot 2p-3=1\cdot 7p$; (ii) $\frac{2}{3}y-8\frac{3}{5}=\frac{2}{15}y-7$.

4. Solve $2v-3w=8$ and $3v+w=1$.

5. In $\triangle$ ABC the altitude AK = 12 cm., and the altitude CM = 15 cm. If BC = x cm. and AB = $(x-5)$ cm., find the length of each.

Test Paper 22.

1. If the vertical angle of an isosceles triangle is $x°$, what is the size of each of the base angles?

2. Divide $a^6 - 1$ by $a^2 - a + 1$.

3. Solve (i) $3{\cdot}7a - 4{\cdot}2 = 6{\cdot}2a + 3{\cdot}3$; (ii) $\frac{1}{2}(3w - 4) = w + 5$.

4. Solve $3w + 7x = 5$ and $6w - 11x = 35$.

5. A rope was cut into two parts, one of which was three times as long as the other. If the difference in their lengths was 10 yd., find what was the length of the whole rope.

Test Paper 23.

1. If each of the base angles of an isosceles triangle is $x°$, calculate the vertical angle. If the vertical angle is three times either of the base ones find it.

2. (i) Simplify $\frac{2}{3}(6x - 9y) - \frac{3}{4}(8x + 4y) - \frac{5}{2}(-12x + 10y)$.

(ii) Multiply $x^3 - 2x^2 - 3x + 4$ by $2x^2 - x - 3$.

3. Solve (i) $7{\cdot}3w - 5{\cdot}2 = 5{\cdot}2w - 9{\cdot}4$;

(ii) $\frac{1}{5}(2y - 3) = \frac{1}{10}y - 6$.

4. If $3x + 4y = 13$ and $x - 3y = 13$, find the value of z when $3x + 8y - 6z = 23$.

5. One number exceeds another by 4. On taking 9 times the smaller from 5 times the greater the result is 8. Find the numbers.

Test Paper 24.

1. The weight of an iron ring is given in lb. by the formula $w = \frac{4}{5}t(R^2 - r^2)$ where t is the thickness and R and r the external and internal radii all in inches. Find w when $t = \frac{1}{2}$, $R = 2\frac{1}{2}$, and $r = 2$.

2. A rectangle is l ft. long and b in. wide. Find its area in sq. yd.

3. Solve for x, $6a + 5x = 20$ given that $3(2a - 1) = 4(a + 3)$.

4. If $4x + 7y = 5$ and $x - 6y = 9$, prove that $6x - 7y = 9x + 2y$.

5. One newspaper costs $\frac{1}{2}$d. more than another. On selling a dozen of each a news-boy draws 3/6. Find the cost of each paper.

Test Paper 25.

1. Find C from the formula $C = \frac{B}{30} \times \frac{1000}{1000 - 2(60 - T)}$ when $B = 31$ and $T = 25$.

2. Divide $2x^4 - 5x^3 + 3x^2 - 6x + 1$ by $x^2 + 4x - 3$.

3. Solve (i) $\frac{3}{y} - 17 = 4$. (ii) $\frac{\sqrt{m}}{3} = -2$.

(iii) $\frac{3}{2}(2v + 6) - \frac{4}{5}(3v + 4) = 3$.

4. If $2y + 5 = 3z$ and $4z - 6 = 3y$, find the value of $7y - 2z$.

5. Nine-tenths of the contents of a cask are run off. Half of what remains is ladled out, three gallons remain. How much did the cask contain at first?

Test Paper 26.

1. If $d = \frac{2(s - an)}{n - 1}$, find d when $s = 124$, $a = 3$ and $n = 12$.

2. A floor is l ft. long and b ft. wide. How many tiles a in. by b in. will be required to cover it?

3. Solve $\frac{5}{6}(4r - 1) - 3 = \frac{3}{4}(2r + 1)$.

4. Solve $3x + 5y = 10$ and $5x + 3y = 8$. If the values of x and y so found also satisfy the equation $3x - 3y = a$, find a.

5. Two numbers differ by 4. If five times the smaller is equal to three times the greater, find the numbers.

Test Paper 27.

1. A grocer has 5 cwt. of tea which he is selling at 3/- a lb. How many lb. has he left after he has sold £x worth of the tea?

2. The sum of $2a^2 - 3a - 4$ and $4a^2 + 7a - 5$ is taken from a certain expression, giving the remainder $3a^2 - 11$. What is the expression?

3. Solve (i) $\sqrt{3x - 2} = 7$. (ii) $16 - \frac{5}{s} = 36$.

(iii) $\frac{2}{7}(v - 1) = 8 - \frac{2}{5}(23 - v)$.

4. Prove that the values of x and y which satisfy the equations $x - 3y + 9 = 0$ and $4x - y - 8 = 0$ also satisfy the equation $2x - 3y + 6 = 0$.

5. The sum of two numbers is 20. If the smaller is x, what is the greater? If $\frac{1}{2}$ of the smaller is equal to $\frac{1}{3}$ of the greater, find the numbers.

Test Paper 28.

1. Evaluate (i) $x^2(6 - x)$; (ii) $x(6 - x)^2$ when $x = -2, -1, 0, 1, 2, 3, 6$.

2. A grocer buys x art. at 5/- per score and sells them at 4/- per doz. Find in pence his total profit.

3. Solve for x (i) $5x - 2a = 10a$.

(ii) $\frac{1}{3}(x - 7) = \frac{1}{2}(3x - 56) - \frac{2}{5}(x + 1)$.

4. If $5y + 8z = 11$; $2y - 3z = -8$ and $xyz = 18$ find x.

5. One newspaper costs 1d. more than another. When a newsagent has sold a dozen of the dearer and 15 of the cheaper, his drawings are 5/6. Find the cost of each newspaper.

Test Paper 29.

1. If a motor car goes x miles on a gallon of petrol, how many gallons will it use in 100 miles. What will be the price in £s of the petrol if a gallon costs 2/6?

2. Multiply $x^2 + 2x - 3$ by $x^2 - x - 20$ and divide the result by $x^2 - 6x + 5$.

3. Solve (i) $mx = -m^3 + m$;

$$\text{(ii) } \frac{1}{2}(3x - 5) - \frac{1}{3}(4x - 7) = \frac{1}{6}(5x - 9).$$

4. If $2p + 3q = 9$ and $3p + 2q = 16$, evaluate $3p - 2q$.

5. A man buys shares of a company at £x each. When their value rises £1 per share he sells them. If the price at which he buys is three-fifths of the price at which he sells, find what he paid per share.

Test Paper 30.

1. Evaluate $\frac{h(h^2 + \frac{3}{4}m^2)}{65}$, when $h = 5$ and $m = 6$.

2. Prove that $(x + 2)^3 = x^3 + 6x^2 + 12x + 8$.

3. (i) Show that $x = 2$ satisfies the equation $x^3 + 5x - 18 = 0$;

(ii) If $\frac{1}{6}(h + 7)$ exceeds $\frac{1}{12}(h + 17)$ by 2, find h.

4. Solve $14x - 15y = 43$ and $12x + 25y = -1$.

5. If 4 be taken from the number n, the result is four-fifths of that obtained by adding 4 to n. Find the value of n.

Test Paper 31.

1. A man walks f ft. in 1 sec. How far will he walk in 1 hr.?
How many miles will he walk in h hr.?
In what time will he walk 1 ft.?
,, ,, ,, ,, 1 yd.?
,, ,, ,, ,, y yd.?

2. Solve $\frac{2}{3}(v - 4) + \frac{5}{6}(3 - 2v) = 3\frac{5}{12}$.

3. If $5a - 3b = -1$ and $10a + 7b = 24$, find the values of $a + b$; $2a + 3b$; $7a - 2b$.

4. Write down the expansions of

$$(y - 3)(y + 8) \text{ and } (x + 6)(2x - 5).$$

5. A gold ring costing £27 is set with 5 diamonds. The gold of the ring is worth £3 less than each of the diamonds. Find the value of the gold and of each of the diamonds.

Test Paper 32.

1. If $y = 3\left(\frac{x}{2} + \frac{2}{x}\right)$, calculate the values of y corresponding to the values 2, $1\frac{1}{2}$, -1, $-\frac{1}{2}$, of x, and tabulate the results.

2. Simplify $\frac{3}{4}(8a - 4b + 12c) - \frac{5}{6}(-6a + 12b - 30c)$.

3. Solve (i) $\frac{20}{w} - 5 = \frac{4}{w} + 3$.

(ii) $\frac{1}{5}(2w - 6) - \frac{1}{15}(4 - 5w) = 5\frac{1}{2}$.

4. (i) Prove $(x + 2)(x - 7) = (x + 6)(x - 11) + 52$ with as little working as possible.

(ii) Write down the expansion of $(2a - 3b)(4a + 5b)$.

5. In a shop there are 840 lb. of tea made up in 2 lb. and 5 lb. packets. If there are 300 packets in all, how many are there of each kind?

Test Paper 33.

1. How many units are there in the number (i) whose tens digit is a and units digit b? (ii) whose hundreds digit is a, tens digit b and units digit c?

2. Solve $\frac{1}{3}(2w - 1) - \frac{1}{8}(3w + 2) = \frac{1}{4}(2 - w)$.

3. If $7l = 2m + 38$ and if l is three times as large as m, find the values of l and m.

4. (i) Simplify $(5a - 2)(a + 3) - (2a + 1)(a - 4)$.

(ii) If $a = b + 4$ express $(a - 6)(a - 2)$ in terms of b.

5. Choose a number, double it and add 27 to what you get. Next multiply the number by 5 and from the product deduct 9. If the second result exceeds the first result by the original number find that number.

Test Paper 34.

1. What must be added to the expression $x^4 - 3x^3 - 7x^2 + 2x + 4$ to make it exactly divisible by $x - 5$.

2. Solve (i) $4{\cdot}7 + 2{\cdot}1s = 7{\cdot}2 - 3{\cdot}6s$ (correct to two decimal places).

(ii) $\frac{4}{5}(2z - 3) - \frac{3}{10}(4z + 1) = \frac{2}{3}(z - 6) + \frac{7}{10}$.

3. If $4s + 3t = 10$ and $6s - 5t = 34$ prove that $s = \frac{20 - 2t}{2 + t^2}$.

4. If $x = 3t - 5$ and $y = 2t + 7$ transform $xy + x - y$ into an expression in t.

5. A boy exchanges his fountain pen for a knife and 2/-, reckoning his pen to be worth half as much again as the knife. What values does he place on the knife and on the pen respectively?

Test Paper 35.

1. If the expression x^3+5x^2-2x+c is exactly divisible by $x-4$, find the value of c.

2. Solve (i) $\frac{l}{x}=\frac{m}{n}$.

(ii) $8{\cdot}6y-3{\cdot}5=6{\cdot}3y-7{\cdot}6$ (correct to two decimal places).

3. If $2x+y-3=3x-y-12=2-x-4y$, find the value of $5x+y$ and prove $\frac{x}{5}+\frac{y}{2}=0$.

4. If $a=x+y$ and $b=x-y$ express $3a-2b$ and $2a+5b$ in terms of x and y. Hence prove $(3a-2b)(2a+5b)=7x^2+32xy-15y^2$.

5. The perimeter of a rectangle is 44 in. If the length were reduced by 3 in. and the breadth increased by 2 in., the length would then be twice the breadth. Find the dimensions of the rectangle.

Test Paper 36.

1. (i) Prove that $x^6-1=(x-1)(x^5+x^4+x^3+x^2+x+1)$.

(ii) If x^3+2x^2-5x-a is exactly divisible by $x+3$, find the value of a.

2. Solve (i) $7{\cdot}3x-4{\cdot}6=6{\cdot}1x-9{\cdot}8$ (correct to two decimal places).

(ii) $\frac{3}{10}(v-3)=2\frac{1}{2}-\frac{1}{3}(2v \quad 13)$.

3. If $3a+4b=-119$ and $5a-11b=102$ prove $a=b$.

4. (i) Solve $(x-3)(x-6)=(x+7)(x-2)$.

(ii) Factorise $a^2+7a-18$.

5. A boy bought x articles at 3d. each and sold the lot at a profit of 5d. If he had bought them for 5d. each and sold them for the same money as before he would have lost 7d. How many articles did he buy?

Test Paper 37.

1. (i) How many dozens are there in a group of x things?

(ii) x articles were bought at 1/- per doz. and sold at 10 a shilling. Find the profit.

2. Simplify $(x+1)(x+2)+(x+3)(x-4)+(x-3)(x-1)$.

3. Solve $x-3y=2x-5y=-2y-2$.

4. (i) Solve $(x+7)(x-2)-(x-8)(x+3)=30$.

(ii) Factorise $m^2+12m-45$.

5. Two numbers differ by 4. If ten times the smaller exceeds three times the greater by 9 find the numbers.

Test Paper 38.

1. (i) Simplify $(x + 6)(x - 3) - (x - 8)(x + 2) + (x - 4)(x - 1)$.
   (ii) Divide $3x^4 + 7x^3 - 13x^2 - 10x + 4$ by $3x^2 - 2x - 4$.
2. Solve $\frac{t - c}{2} - \frac{t - 2c}{3} = 2c$ when (i) $c = 4$, (ii) $c = -4$.
3. Solve $3x - 4y = 27$, $4x + 5y = 5$, $6x + 3y - 2z = 13$.
4. (i) Solve $(x + 6)(x - 2) = 2x^2 - (x + 5)(x - 8)$.
   (ii) Factorise $t^2 - 19t + 48$ and $w^2 - 49$.
5. An oblong piece of wood is cut through, along a line parallel to its shorter edge, so that it is divided into a square and rectangle. The length of the rectangle is 2 in. greater than the side of the square. Find the dimensions of the rectangle if it is 10 sq. in. larger than the square.

Test Paper 39.

1. Compute the following bill in pence :—

   x articles at 7d. per doz.
   $2x$ „ „ 1/3 per score
   $3x$ „ „ 1/- for 15.

2. Simplify (i) $2(x + 6)(x + 3) + 3(x - 2)(x + 4)$.
   (ii) $7(x - 3)(x + 5) - 4(x - 1)(x + 8)$.
3. Solve $\frac{r}{3} + \frac{s}{2} = 1$ and $\frac{r}{4} - \frac{s}{6} = 1\frac{5}{6}$.
4. (i) Solve $(2t - 3)(4t + 1) - 4t(2t - 5) = 9$.
   (ii) What factor is common to each of the expressions

   $$y^2 + 2y - 15 \text{ and } y^2 - 9?$$

5. I bought x articles at 9d. per doz. and sold them at 1/10 per score. By so doing I gained 1/9. How many articles did I buy?

Test Paper 40.

1. Simplify $5(m + 4)(m - 6) - 3(m + 8)(m - 2) - (m - 6)(m - 9)$.
2. Solve for t the equation $\frac{t - 2c}{3} - \frac{t - c}{2} = 3c$.
3. Solve $\frac{3x + 2}{5} - 2y = 5$ and $4x + 3y + 2 = 0$.
4. (i) Factorise $y^2 - 5y - 66$ and $z^2 + 13z + 42$.
   (ii) Solve $x^2 - 3 = 13$ and $t^2 + 28 = 11t$.
5. If one side of a square be extended 4 in. and an adjacent side shortened by 3 in. then a rectangle will be obtained equal in area to the square. Find the dimensions of the rectangle.

Test Paper 41.

1. Evaluate the formula $\frac{a}{3}(x^2+\sqrt{xy}+y^2)$ when $a=15$, $x=8$, and $y=2$.

2. (i) Simplify $(2a-3)^2-4(a+3)(a-3)$.
(ii) Factorise $6y^2-7y-5$.

3. (i) Solve $x^2-5x=0$; (ii) $s^2+15s=34$.

4. Factorise (i) a^2x^2-5ax; (ii) $9a^2-49b^2$; (iii) $3p^2-75q^2$.

5. The sides of a square are each x in. long. One side is extended 2 in. and an adjacent side is extended 1 in. In this way a rectangle is formed which is 50 sq. in. greater than the square. Find the length of a side of the square.

Test Paper 42.

1. When $x=4$ then $px+q=13$. If the value of $px+q$ is 25 when $x=7$, find p and q.

2. (i) Prove by the use of factors that

$$(x-4)(x^2+x-2)=(x-1)(x^2-2x-8).$$

(ii) Simplify $\frac{2x^2+5x-3}{2x^2+9x-5}$.

3. Prove that the equations $2z^2+5z=12$ and $2z^2+3=5z$ have a common root.

4. Factorise (i) $3bc-6b^2c^2$; (ii) $16m^2-49p^2q^2$; (iii) $(2l-5)^2-4$.

5. Four years ago a father was three times as old as his son, and eight years hence the father will be twice as old as the son. Find the present age of each.

Test Paper 43.

1. A room is l ft. long by b ft. broad. The carpet leaves a margin 1 ft. wide all round, which is stained. Find the area of the stained part.

2. Solve $\frac{x+1}{2}=\frac{y+7}{3}$ and $\frac{2x-1}{3}-\frac{y}{6}=\frac{2+y}{6}$.

3. (i) What factor is common to the following expressions $2y^2-18$; $4y^2+12y-72$; $2y^3+4y^2-30y$?
(ii) Simplify $\frac{2a^2-72}{2a^2-6a-108}$.

4. Find the sum, difference, and product of the roots of the equation $8x^2-10x+3=0$.

5. 4 sheep and 3 cows cost £56. If 12 sheep and 6 cows cost £136, find the cost of 5 sheep and 2 cows.

Test Paper 44.

1. If $y = mx + b$, find the value of x when $y = 8$ given that $x = 2$ when $y = 3$, and that $x = -4$ when $y = 1$.

2. Show that $x - 4$ is a factor of $2x^3 - 7x^2 - 10x + 24$. Find the other factors.

3. (i) Solve $6z^2 - 117 = z$.

(ii) Solve $w(2x - 3) = 7$ given that $x = 2w$.

4. (i) Factorise $a(x + y) - 2b(x + y)$ and $(7b - 2c)^2 - (3b + 5c)^2$.

(ii) Simplify $\dfrac{4y^2 - 9z^2}{2xy + 3xz}$.

5. The value of 52 coins consisting of shillings and threepences, is £1 11/-. How many coins of each kind are there?

Test Paper 45.

1. A grocer buys p stones of cheese at x/- per stone and sells it at x pence per lb. Show that his profit is $2px$ pence. Use the formula to find the profit on 2 cwt. of cheese bought at 10/- per stone and sold at 10d. per lb.

2. Solve (correct to two places of decimals)

$$\frac{1}{2}(3{\cdot}1x - 2{\cdot}9) = \frac{1}{4}(4{\cdot}1x - 1{\cdot}4).$$

3. Solve $x^2 - y = 7$ and $2x + y = 17$.

4. If $y = ax^2$ and $Y = aX^2$, prove $\dfrac{y - Y}{x + X} = a(x - X)$.

5. The handle of a toasting fork is in three parts which "telescope" into each other. The two disappearing parts are equal, and together are 6 inches longer than the third part. When only one disappearing part is extended the handle of the fork is 21 inches long. Find the full length of the handle.

Test Paper 46.

1. Solve (i) $\dfrac{1}{2x - 3} = \dfrac{1}{9}$. (ii) $\dfrac{4}{3x - 5} = \dfrac{2}{3}$.

2. Solve $5x - 6y = 13$ and $15x + 11y = 1$, giving the answers correct to two places of decimals.

3. Show that $s + 2$ is a factor of $3s^3 + 2s^2 - 12s - 8$ and find the other factors.

4. (i) Simplify $\dfrac{6x - 3y - 7}{4x - 2y + 4}$ given that $y = 2x - 3$.

(ii) Factorise $(p^2 + q^2)^2 - p^4$.

5. The membership of a tennis club was raised to 80 by the admission of new members at an annual subscription of 47/6 each.

If the subscription for old members is 37/6 and the total income is £165, how many new members joined?

Test Paper 47.

1. The diagram shows a movable electric lamp L suspended at the end of a sliding cord, and balanced by a weight W. If the distances of W from A and B be x inches, and if the lamp be hanging y inches from the ceiling, find the relation between x and y given that the length of the whole cord is l inches.

2. Solve the following equations :—

$$\frac{2}{3}(4x - y) - \frac{3}{4}(2x - 3y) = 2.$$

$$\frac{4}{5}(4x + 7y) - \frac{3}{10}(9y - 2x) = 2.$$

3. (i) Solve $(t + 7)(t + 8) = 72$.

(ii) Form the quadratic equation whose roots are 2 and -3.

4. Factorise (i) $21 - 4t - t^2$; (ii) $y(y + 24) - 3(y + 24)$; (iii) $(9a - 3b)^2 - (5a + 7b)^2$.

5. A and B play cards for money, A having 10/- more than B. When A wins 3/4 he has twice as much money as B. How much has each at first?

Test Paper 48.

1. A dealer buys coal at £x per ton and sells it at y/- per cwt. If he gains 20 per cent. prove $6x = 5y$.

2. (i) Solve $\frac{2}{3}(4s - 1) - \frac{5}{6}(s + 2) = \frac{8}{9}(2s - 1) - \frac{1}{2}$.

(ii) If $P(W + w) = 2Ww$ find w when $P = 5\frac{1}{3}$ and $W = 8$.

3. If $y = 2x + 3$ express $x^2 + y^2$ in terms of x. For what values of x will the result be 2?

4. Factorise (i) $2a^3 - 18a$; (ii) $(3a - 2)^2 - (3b - 2)^2$.

5. There are 50 coins in a bag consisting of pence, sixpences, and shillings. The number of pence is twice as great as the number of shillings and the total value of the money is £1 1s. 4d. How many coins of each kind are there?

Test Paper 49.

1. Fill up the gaps in the following table (calculating correctly to the first decimal place), given that $4x + 3y = 10$:—

x	0		1		2		-1		-2		$\frac{1}{4}$	
y		0		1		2		-1		-2		$-\frac{1}{3}$

2. Multiply $x^3 - x^2 + x - 2$ by $x^2 - x + 3$, and divide the result by $x^2 + 3x - 2$.

3. The equation $y = mx + b$ is satisfied (i) when $x = 2$ and $y = 3$; (ii) when $x = 3$ and $y = 2$. Find the value of y when $x = -10$.

4. (i) Factorise $81 - 16m^4$ and $(x + 2y)^2 - (x - y)^2$.

(ii) Simplify $\dfrac{a^2x^2 - ax}{b^2x^3} \times \dfrac{b^4x^2}{a^4x^3 - a^3x^2}$.

5. The base of an isosceles triangle is 8 in. shorter than the other two sides taken together. If twice the base is 1 in. shorter than either of the equal sides, find the lengths of the sides of the triangle.

Test Paper 50.

1. Along a mile of straight road n telegraph poles are placed at equal intervals. Find in yards the length of the interval.

2. (i) If $(a + b)(a - c) + (a - b)(a + c) = 0$, prove $a^2 = bc$.

(ii) For what value of h will the two expressions $\frac{3}{4}(3h + 4)$ and $\frac{2}{3}(2h + 1)$ differ by 6, the first being the greater.

3. The equation $t^2 + at = a^2 + 1$ is satisfied when $t = -5$. Find what values a can have.

4. Factorise $9x^2 - \frac{1}{4}y^2$ and $9(2x - 3)^2 - 4x^2$.

5. A sum of £2 15s. is made up of half-crowns and florins. If the number of half-crowns be halved and the number of florins doubled the value of the money would be £2 2s. 6d. How many coins of each kind are there?

Test Paper 51.

1. (i) A wheel revolves n times in a mile. Find its radius in feet.

(ii) If $(a + b)^2 - (a - c)^2 = a(a + 2b + 2c)$ prove that a, b, and c are the sides of a right-angled triangle.

2. For what values of h will the product of the two expressions $\frac{3}{4}(3h + 4)$ and $\frac{2}{3}(2h + 1)$ be equal to 7?

3. Factorise (i) $x^3 - x^2 - 2x$; (ii) $2v^2 - v - 10$.

4. Simplify (i) $\dfrac{2a^2 - 3a - 2}{a - 2}$ and (ii) $\dfrac{2x - 1}{2x^2 - 3x + 1}$.

5. The sum of a certain positive number and its square is 20. Find the number.

Test Paper 52.

1. A grocer buys a ton of a certain commodity for £C and sells it at d pence per lb. Find his total gain in £. If by selling the commodity at 6d. per lb. he clears a profit equal to the cost, deduce from the formula just established, what he paid for the ton.

2. (i) Solve $2m(m - 3) = 17(3 - m)$.

(ii) If $s = \dfrac{n}{2}(3n + 1)$, find n when (1) $s = 26$; (2) $s = 5n$.

3. Factorise (i) $a^2(2x - y) - b^2(2x - y)$; (ii) $(2x^2 - x - 1)^2 - 4$.

4. Simplify $\dfrac{x^2 - 4}{x^2 - 3x - 10} \times \dfrac{2x^2 - 10x}{x^2 + 2x - 8}$.

5. The area of a rectangle of perimeter 20 yd. is 144 sq. ft. Find the dimensions of the rectangle.

Test Paper 53.

1. If $2x - 3y = 2$, $7x - 10y = 8$, and $x^2 + y^2 + z^2 = 45$, find z.

2. (i) Solve $2t(t + 5) = 4 - (t + 2)(t - 2)$.

(ii) If $x - ty + at^2 = 0$, find t when $x = y = 4a$.

3. (i) Factorise $x(2a - 3) - y(2a - 3) + 4a - 6$.

(ii) In a right-angled triangle the shortest side is 8 cm. and the sum of the other two sides is 32 cm. Find how much longer is the hypotenuse than the remaining side.

4. Simplify $\dfrac{a^2 - 5a + 4}{a - 3} \times \dfrac{a^2 - 5a + 6}{a^2 - 3a + 2}$.

5. The product of two consecutive numbers exceeds the difference of their squares by 1. Find the numbers.

Test Paper 54.

1. If $x^2 + y^2 - 4x - 6y + 12 = 0$, find y when $x = y$.

2. (i) Factorise $(t + 4)^2 - (t + 4) - 12$.

(ii) Prove that $(a^2 + 1)$, $(a^2 - 1)$, and $2a$ are the sides of a right-angled triangle.

3. Simplify $\dfrac{2a^2 - 3a + 1}{16a^4 - 81} \div \dfrac{7a^2 - 7a}{2a - 3}$.

4. (i) Find the L.C.M. of a^2; $ax - ab$; and $x^2 - b^2$.

(ii) Simplify $\dfrac{2w - 4}{3} + \dfrac{5w + 7}{6}$.

5. One side of a square is extended 2 in., and an adjacent side 5 in. and so a rectangle can be formed whose area is 40 sq. in. Find the side of the square.

Test Paper 55.

1. A man buys 3 oz. of tobacco at b pence per oz., and c cigarettes at 10 for 6d. paying in all a/-. Prove $20a = 5b + c$.

2. Factorise (i) $(x+2)(x-3) - 4(x+2)$.

(ii) $36(3a-b)^2 - 25(2a+3b)^2$.

3. Simplify $\dfrac{9x^2-49}{4x^2+x-5} \times \dfrac{x^2-1}{3x^2-4x-7}$.

4. (i) Find the L.C.M. of $x^2 - xy$; $xy - y^2$; and $x^3y - xy^3$.

(ii) Simplify $\dfrac{3}{w} + \dfrac{4}{w-1}$.

5. A lidless box is 1 ft. deep and is 2 ft. longer than it is wide. If the area of the bottom is 4 sq. ft. more than the area of the sides, find the length and breadth of the box.

Test Paper 56.

1. (i) Solve $2t - 1 = \dfrac{3}{t}$.

(ii) If $a = b(c^2-1) + c(b^2-1)$, find c when $a = 0$ and $b = 6$.

2. Simplify $\dfrac{2a^2-10a}{a^2+4a-21} \times \dfrac{a^2+2a-15}{a^2-25}$.

3. Factorise (i) $x^2 + 3ax - 28a^2$.

(ii) $(3m^2+4m-3)^2 - (2m^2-3m-3)^2$.

4. Find the L.C.M. of c^2-4, c^2+4c+4, and c^2-4c+4.

5. By extending the sides of a cube 2 ft., its volume will be increased 152 cub. ft. Find the length of a side of the cube.

Test Paper 57.

1. If $s = \dfrac{3t}{t+6}$ calculate s when $t = -2, -1, 0, 4$, and 9. Find t when $s = 5$. When do s and t have the same value?

2. Factorise (i) $(x^2 - 3x - 45)^2 - 4(x^2 - 3x - 45) - 45$.

(ii) $16(5a - 4b - 2c)^2 - 9(a + 3b - c)^2$.

3. Simplify $\dfrac{4a^2 - 9}{10a^2 - 15a} \div \dfrac{2a^2 - 9a - 18}{a^3 - 6a^2}$.

4. Find the L.C.M. of $100y^2 - 16$, $30y^3 + 12y^2$, $225y^2 - 180y + 36$.

5. A rectangular plot of ground is $2b$ ft. long and b ft. broad. Round it runs a path c ft. wide. Prove that the area of the path is $2c(3b + 2c)$ sq. ft. If the breadth of the plot is 20 ft. and the area of the path 396 sq. ft., find the width of the path.

Test Paper 58.

1. If $x = \dfrac{ad}{c - d}$ express x in terms of d given that $a = 1$ and $c + d = 40$. Find d when $x = 24\frac{1}{2}$.

2. If d be the number of diagonals of an n-sided figure then $d = \frac{n}{2}(n - 3)$. What figure has 12 more diagonals than it has sides?

3. Simplify $\dfrac{18a^2 - 12a}{9a^2 - 4} \times \dfrac{3a^2 - 16a - 12}{2a^2 - 72} \div \dfrac{3a^3 - 15a^2}{a^2 + a - 30}$.

4. Find the L.C.M. of

$$c^2 + c - 2\,;\ 4c^2 + 7c - 2\,;\ \text{and } (6c - 3)^2 - (2c + 1)^2.$$

5. The sum of two numbers is 15 and the difference of their squares is 15. Find the numbers.

Test Paper 59.

1. If the area of a rectangle be a sq. in. and the length be l in., work out an expression for the perimeter. If the area is 45 sq. in. and the perimeter is 19 in. more than the length, find the dimensions of the rectangle.

2. (i) Solve for t the equation $6t^2 + tw - 15w^2 = 0$.

(ii) Form the quadratic equation whose roots are $\frac{1}{2}$ and $\frac{1}{3}$.

3. Factorise $16(4x - 3)^2 - 9(2x + 1)^2 - 5(4x^2 - 9)$.

4. Simplify $\dfrac{(k+l)^2 - m^2}{(k+m)^2 - l^2} \times \dfrac{(k-l)^2 - m^2}{(k-m)^2 - l^2}$.

5. I can walk a certain distance at 4 m.p.h., in 10 min. less than at 3 m.p.h. What is the distance?

Test Paper 60.

1. A can do a piece of work in a days and B can do it in b days. What fraction of the work will they do per day working together? How long will they take to do the whole job?

2. If $x^2 - y^2 = 60$ and $x + y = 10$, find the value of $x - y$, and hence find the values of x and y.

3. Simplify $\dfrac{2}{a - 2} - \dfrac{3}{a^2 - 4}$.

4. Solve $y^2 = 4y + 11$ giving the roots correct to two places of decimals.

5. I walk from my home to the station at 4 m.p.h. and am 3 min. too early for the train. If I walk at 3 m.p.h. I am 1 min. late How far do I live from the station?

Test Paper 61.

1. (i) Solve for u the equation $\frac{1}{u} + \frac{1}{v} = \frac{1}{6}$. Find u when $v = 7{\cdot}5$.

(ii) Solve for z the equation $az - a^2 = bz - b^2$.

2. Find the L.C.M. of $x^2 + 6ax + 9a^2$, $x^2 - 6ax + 9a^2$, and $x^4 - 81a^4$.

3. Simplify $\frac{3c - 11}{c^2 - c} - \frac{11}{c} + \frac{3}{c - 1}$.

4. Solve (correct to two decimal places) the equation

$$2x^2 - 10x + 11 = 0.$$

5. One train travels 12 m.p.h. faster than another. In 3 hr. the faster reaches its terminus, and in 5 hr. the slower is 36 miles beyond where the faster stops. Find the speeds of the trains.

Test Paper 62.

1. Solve for p and q the equations

$$p - q = a \quad \text{and} \quad p - cq = b.$$

2. Simplify $\frac{x^4 - 16a^4}{a^2x^2 - 4a^3x + 4a^4} \div \frac{x^2 + 4ax + 4a^2}{a^2}$.

3. Simplify (i) $h - \frac{(3a + b)h}{4(a + b)}$; (ii) $\frac{5a + 3}{6a^2 - 5a - 6} - \frac{1}{2a - 3}$.

4. (i) Solve (correct to two decimal places) $x = 3 + \frac{1}{x}$.

(ii) The equation $x^2 + px + q = 0$ is satisfied when $x = 2$ and $x = -5$. Find p and q.

5. The area of a quadrilateral ABCD is 70 sq. in. Of the perpendiculars drawn from A and C to BD, one is 2 in. shorter than, and the other is 4 in. shorter than, BD. Find BD.

Test Paper 63.

1. Solve for x the equation $xy = by - ax - ab$.

2. Simplify $\frac{(x^2 + 5xy - 2y^2)^2 - (x^2 + 3xy + 2y^2)^2}{x^2y - 2xy^2}$.

3. Simplify $\frac{1}{x + y} - \frac{2x}{x^2 - y^2} - \frac{1}{x - y}$.

4. Solve (correct to two places of decimals) $2x^2 = 5 - x$.

5. A cylinder is 3 ft. deep and has one end closed. From the centre of the end plate a circular lid 8 ft. in diameter can be removed, so leaving a portion equal in area to the curved surface of the cylinder. Find the radius of the cylinder.

Test Paper 64.

1. A hemispherical lid fits on to the mouth of a cylindrical tank 4 ft. wide and h ft. deep. If the volume of the cylinder is 6 cub. ft. more than that of the lid prove that $h = \frac{8\pi + 9}{6\pi}$.

[Volume of a sphere of radius $r = \frac{4}{3}\pi r^3$.]

2. Solve for x the equation $ax + 2ab = a^2 + bx + b^2$.
3. Find the L.C.M. of $2c^2 + c - 6$; $4c^3 - 9c$; $4c^2 + 12c + 9$.
4. Simplify $\frac{4}{a-1} - \frac{9a+7}{3a^2 - 2a - 1}$.
5. Solve (correct to two decimal places) the equation $2 + \frac{5}{x} = \frac{6}{x^2}$.

Test Paper 65.

1. It costs £6 to paper the walls of a room l ft. long and b ft. broad at a pence per sq. ft. Find the height of the room.
2. Solve $y = ax + b$ and $y = mx$.
3. (i) Factorise $(x + 4)(x + 3)(x - 1) - (x + 3)(x + 2)(x - 1)$.
   (ii) Prove that $(x^2 + y^2)^2 - x^2y^2 = (x^4 + x^2y^2 + y^4)$. Hence find the factors of $x^4 + x^2y^2 + y^4$.
4. Simplify $\frac{2a}{a^2 - y^2} + \frac{2a}{a - y} - \frac{1}{a + y}$.
5. (i) Solve (giving the roots correct to two decimal places) $5x(x + 2) = 8 - x$.
   (ii) The equation $x^2 - (4a + 3)x + (a + 3)^2 - 1 = 0$ is satisfied when $x = 3$. Find a and the remaining root.

Test Paper 66.

1. A grocer mixes 60 lb. of tea at a pence per lb., with 80 lb. of tea costing 6d. per lb. more. At how much per lb. must he sell the mixture if he is to make a profit of £2 13s. 4d.?
2. (i) Solve $(x - a)(x + a) = x(x - a) + b(x - b)$.
   (ii) If $x = at^2$ and $y = 2at$, find t given $ax + 3by = 7b^2$.
3. If $u = ax + by$ and $v = bx + ay$ simplify

   (i) $\frac{au - bv}{a - b}$; (ii) $\frac{au - bv}{av - bu}$.

4. Simplify $\frac{p + 2}{p^2 - 5p + 6} + \frac{5p - 6}{p^2 - 3p + 2}$.
5. The distance between the parallel sides of a trapezium is 3 in. more than the length of the shorter of these sides and 5 in. less than the length of the greater. If the area of the trapezium is 56 sq. in., find the lengths of the parallel sides.

Test Paper 67.

1. If t tons of goods are carried m miles for £a, how far will w tons be carried for £b?

2. (i) Solve $ax + by = 1$; $x - y = 3$.
   (ii) If $5xy = 5x + 6y$, find the values of x and y given that $x = t$ and $y = \dfrac{5}{t}$.

3. If $x = t^2 + t - 1$ and $y = t^2 - t - 1$
   (i) Prove $x^2 - y^2 = 4t(t + 1)(t - 1)$.
   (ii) Simplify $\dfrac{x + y}{x - y} \times \dfrac{x + y + 2}{x - y - 2}$.

4. Simplify $\dfrac{3c + 5}{c^2 + 4c + 3} - \dfrac{1}{c + 3} - \dfrac{1}{c + 1}$.

5. In a trapezium one of the parallel sides is 3 in. longer than the other. One of the non-parallel sides is perpendicular to the parallel sides and is equal to the shorter of these. If the area of the trapezium is 22 sq. in. find the perimeter.

Test Paper 68.

1. If $\dfrac{a + x}{a - x} = \dfrac{y - b}{y + b}$ prove $xy + ab = 0$.

2. Factorise (i) $x^3 + x^2 + x + 1$ and $8x^3 + 27y^3$.
   (ii) $9x^4 - 18x^2y^2 + 9y^4$.

3. Simplify $\dfrac{x}{x^2 - 1} - \dfrac{1}{x + 1} - \dfrac{2}{1 - x}$.

4. Solve $\dfrac{2}{x - 1} = \dfrac{3}{x + 1} - \dfrac{3}{x}$.

5. The altitude of an isosceles triangle is 2 in. shorter than the base and 1 in shorter than either of the equal sides. Find the lengths of the sides of the triangle.

Test Paper 69.

1. If $x = t - 1$ and $y = \dfrac{2}{t + 1}$, prove $xy = 2(1 - y)$.

2. Solve (i) $\dfrac{x + 2a}{x - a} = \dfrac{x + 3a}{x - 2a}$.
   (ii) $\dfrac{5}{x + 2} = \dfrac{14}{x + 4} - \dfrac{3}{x}$.

3. Factorise (i) $\dfrac{1}{9x^2} - \dfrac{1}{4y^2}$, (ii) $\dfrac{20}{a^2} + \dfrac{1}{a} - 1$, (iii) $a^3 + \dfrac{27}{a^3}$.

4. Simplify (i) $\dfrac{p + q}{p - q} - \dfrac{p^2 + q^2}{p^2 - q^2}$ and (ii) $\dfrac{p - q}{p + q} - \dfrac{p^2 + q^2}{p^2 - q^2}$.
   Show how to deduce the answer to (ii) from that of (i).

5. It costs £3 7s. 6d. to paint the walls of a room 12 ft. wide at 1½d. per sq. ft. If the length is 5 ft. more than the height, find the dimensions of the room.

Test Paper 70.

1. If $x=t^2-1$ and $y=t+1$, prove (i) $y^2=x+2y$,
(ii) $(x+y)^2=y^2(1+x)$.

2. Factorise (i) $x^5-4x^3-8x^2+32$.
(ii) $x(y-z)+(y-z)(y+z)$.

3. Simplify $\dfrac{2a+b}{2a-b}-\dfrac{2a-b}{2a+b}+\dfrac{6a^2b+2ab^2}{(4a^2-b^2)(3a+b)}$.

4. (i) Solve $ax-by=c$, $bx-ay=0$.
(ii) If $xy=48$ and $x=y+2$, find x and y.

5. X had £8 more than Y. X gave Y $\frac{1}{12}$ of his money and then they had equal sums. What did each have at first?

Test Paper 71.

1. A plot of ground is b ft. square. Round the outside runs a path c ft. wide. Prove that the area of the path is $4c\ (b+c)$ sq. ft. If the area of the path is $\dfrac{9}{16}$ of the area of the square, prove $c=\dfrac{b}{8}$.

2. Solve for x the equation $\dfrac{x}{a}+\dfrac{x-3}{a-2}=a+2$.

3. Factorise (i) y^3+4y^2-21y, (ii) $(2x+3y)^2-4c^2$, (iii) x^3y^3+64, (iv) $z^3-b^2z-cz+bc$.

4. Simplify $\dfrac{\dfrac{10-t}{5}+\dfrac{t}{5-t}}{\dfrac{10-t}{5-t}-\dfrac{t}{5}}$.

5. Solve $\dfrac{2x+1}{x-1}-\dfrac{2x-7}{x-3}=2$.

Test Paper 72.

1. Simplify the expression $s^2+(s-a)^2+(s-b)^2+(s-c)^2$ arranging it in descending powers of s.
If $2s=a+b+c$, prove that the expression is equal to $a^2+b^2+c^2$.

2. Solve for z the equation $\dfrac{z-a}{a-b}-\dfrac{z-b}{a+b}=\dfrac{a^2+b^2}{a^2-b^2}$.

3. Simplify (i) $\left(1-\dfrac{2b^3}{a^3+b^3}\right)\left(a+\dfrac{b^2}{a-b}\right)\div\left(b+\dfrac{a^2}{a+b}\right)$.

(ii) $\dfrac{x-1}{x^2+5x+6}+\dfrac{x+2}{x^2+4x+3}-\dfrac{2(x-3)}{x^2+3x+2}$.

4. Solve $\dfrac{x+2}{x-1} - \dfrac{1}{3-x} = 3$.

5. A number consists of two digits whose sum is 12. If the digits be reversed the number is diminished by 18. Find the number.

Test Paper 73.

1. Simplify $\dfrac{s^2 - (s-a)^2}{(s-b)^2 - (s-c)^2}$ given that $2s = a + b + c$.
2. Solve for y the equation $rs(y^2 - 1) = (r - sy)(s - ry)$.
3. Simplify $\left(\dfrac{m}{m-2} + \dfrac{5}{m-8}\right)\left(\dfrac{m-3}{3m-8} - \dfrac{2}{m+2}\right)$.
4. Find the L.C.M. of $a^2 - 4$; $2a^2 + 3a - 2$; $2a^2 - 3a - 2$; $a^3 + 8$.
5. Solve $\dfrac{t+4}{2t} + \dfrac{t+2}{1-t} + 1\frac{1}{2} = 0$.

Test Paper 74.

1. Solve the equation $A = P\left(1 + \dfrac{rn}{100}\right)$ for P and n in turn.
2. Factorise (i) $(p+q)(r-s) - (p-q)(r-s)$.
   (ii) $15ax - 2by + 3bx - 10ay$.
   (iii) $(2a^2 + a + 1)^2 - (a^2 - 3a + 4)^2$.
   (iv) $x^4 - 29x^2 + 100$.
3. If $l = \dfrac{b+c}{1-bc}$ and $m = \dfrac{b-c}{1+bc}$, simplify $\dfrac{l+m}{1-lm}$.
4. Solve (correct to two decimal places) the equation

$$\frac{x+2}{x-1} = \frac{6}{x-2}.$$

5. A tank can hold 240 gallons of water. A supply pipe takes 5 minutes longer to fill it when a waste pipe capable of running off 4 gallons per minute is open, than what it does when the latter pipe is closed. Find the rate of flow of the supply pipe.

Test Paper 75.

1. From the four corners of a rectangular sheet of tin, square pieces are cut out, and the wings are then bent up so as to form a tobacco box l in. long, b in. wide, and x in. deep. Prove that the area of the original sheet was $[4x^2 + 2(l+b)x + lb.]$ sq. in.

2. (i) Solve for x the equation $\dfrac{x}{a} - \dfrac{x-2}{a-2} = \dfrac{2a}{a^2-4}$.
   (ii) Solve $px + qy = p^2$, $qx + py = q^2$.

3. Simplify $\dfrac{(px + qy)^2 - (px - qy)^2}{(rx - sy)^2 - (rx + sy)^2}$.

4. Simplify $\dfrac{2x + 1}{2x^2 + 3x + 1} - \dfrac{2x - 1}{2x^2 - 3x + 1} + \dfrac{2x^2}{x^2 - 1}$.

5. In Example 1 of this paper, if the box is 6 in. longer and 4 in. wider than it is deep, and if the area of the sheet was 120 sq. in., find the depth of the box.

Test Paper 76.

1. Prove that the sum of the squares of any three consecutive even numbers exceeds one-third of the square of the sum of the numbers by 8.

2. Solve $\dfrac{b}{r} + t = f$, $\dfrac{b}{r} - t = g$ for r and t.

3. Factorise (i) $(a + b + 1)(a - b - 1) + 4b$.
(ii) $z^3 + z^2 - z - 1$.

4. Simplify $\dfrac{a^2 + ab}{a^2 + 2ab + b^2} + \dfrac{a^2 - ab}{a^2 - 2ab + b^2} - \dfrac{a^4 + a^2b^2}{a^4 - b^4}$.

5. When $x = 0$ find the value of $\dfrac{x - 4}{x - 1} - \dfrac{2x}{3x + 2}$. For what other value of x does the expression have the same value as before?

Test Paper 77.

1. One clock gains 1 min. per day while another loses b min. per day. If they originally indicate the same time, after how many days will they be c min. apart? How long will it be before they again read alike?

2. Solve (i) $\dfrac{3}{s - \frac{2}{3}} = \dfrac{1}{s - \frac{2}{5}}$;

(ii) $y + \dfrac{2}{z} = 9$; $y - \dfrac{1}{z} = 13$.

3. (i) If $x = at^2 + bt + c$ and $y = ct^2 + bt + a$, express $x - y$ as a product of factors.
(ii) Factorise $4t^2 + 2(a + b)t + ab$.

4. Simplify (i) $\left(x + 3 - \dfrac{6}{x - 2}\right)\left(x + 6 + \dfrac{8}{x - 3}\right)$;

(ii) $\dfrac{x^3 - 1}{x^2 + y^2 + 1}$ given that $y = x + 1$.

5. Solve $\dfrac{x + 1}{x - 4} - \dfrac{x + 3}{x - 3} = \dfrac{6}{x - 2}$.

Test Paper 78.

1. If $x = \frac{(1 - t^2)a}{1 + t^2}$ and $y = \frac{2tb}{1 + t^2}$, prove $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

2. Solve (i) $\left(x + \frac{a}{b}\right)\left(x + \frac{b}{a}\right) + \left(x - \frac{a}{b}\right)\left(x - \frac{b}{a}\right) = 4x$.

(ii) $x - \cdot 5y = 2(x - \cdot 6)$
$x - 3\cdot 7 = 2(y + 1)$.

3. (i) Factorise $(x + y + 4)(x + y - 4) - 4xy$.
(ii) Divide $(ac + bd)^2 + (ad - bc)^2$ by $a^2 + b^2$.

4. Solve $\frac{1}{2(x - 1)} + \frac{3}{x^2 - 1} = \frac{1}{4}$.

5. A man bought some shares in one company at £5 each, and some in another company at £7 each, paying in all £310. The former rose in price £2 per share, while the latter fell £3 per share, with the result that his capital depreciated £50. How many shares did he buy in each concern?

Test Paper 79.

1. Show that the square of the sum of any two consecutive even numbers is four times the square of the odd number lying between them.

2. Solve $b(ax + b) = b(y - ax) = 2by - a^2$.

3. Simplify $\left(\frac{3a + 2x}{3a - 2x} + 1\right)\left(\frac{3a - 2x}{3a + 2x} + 1\right) + 4\left(1 - \frac{9a^2}{9a^2 - 4x^2}\right)$.

4. Factorise (i) $x(x^2 - a^2) - (x - a)(x^2 + ax + a^2)$.
(ii) $(b + c + 1)^2 - (b + c + 1)(b - c - 1)$.

5. For two values of z the expression $\frac{(z + 4)(z - 3)}{(z - 2)(z + 3)}$ is the same. If one of the values of z is 0, what is the other value?

Test Paper 80.

1. Water stands in a tank of base a sq. ft., to a depth of b ft., and in a tank of base b sq. ft. to a depth of a ft. If the two tanks are put in connection by a very short pipe so that the water settles at a level of l ft. in both, show that

$$l = \frac{2ab}{a + b}.$$

If $l = \frac{a + b}{2}$, prove $a = b$.

2. If $br^2 + \frac{1}{t} = f$ and $br^2 - \frac{1}{t} = g$, prove $\frac{r^2}{t} = \frac{f^2 - g^2}{4b}$.

3. Find (i) the sum of the roots; (ii) the sum of the products of the roots in pairs; (iii) the product of the roots of the equation $x^3 - 14x^2 + 49x - 36 = 0$, it being given that $x - 1$ is a factor of the L.H.S. Compare your results with the coefficients of the expression.

4. Simplify $\dfrac{3x - 3}{2x^2 - 6x + 4} - \dfrac{x - 2}{x^2 - 4} - \dfrac{x + 10}{2x^2 + 8}$.

5. In Example 1 of this paper the water is 2 ft. deeper in the first tank than in the second, and settles at a level of $1\frac{1}{2}$ ft. in each. Find its original depth in each tank.

Test Paper 81.

1. If $a = \dfrac{x^2 - y}{1 - xy}$ and $b = \dfrac{y^2 - x}{1 - xy}$, show that
$$a + bx + y = b + ay + x = 0.$$
Hence find x and y in terms of a and b.

2. Factorise (i) $y^4 - 5y^3 + 20y - 16$.
(ii) $(a + 1)(a^2 - 2a - 8) + (a + 2)(a^2 - a - 2)$.

3. Solve $\dfrac{5}{t + 1} - \dfrac{1}{t + 7} = \dfrac{3}{t - 1}$.

4. Simplify $\dfrac{1}{2(x - 1)^2} - \dfrac{1}{3(x - 1)} + \dfrac{2x + 3}{6(x^2 + 2x + 3)}$.

5. In a certain workshop, a workman is allowed a certain time, C hr., in which to do a job. If he does it in less time he is allowed a bonus of B pence calculated from the formula $B = \dfrac{KW(C - K)}{C}$, where K is the time in hours in which the job is done and W is the number of pence paid per hour. Find K when B = 81, W = 18, and C = 24, given that the work cannot be done within 12 hours.

Test Paper 82.

1. If $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{1}{20}$ and if $d = u + v$, prove $d = \dfrac{u^2}{u - 20}$.
Find u when $d = 80$.

2. (i) Solve $\dfrac{5 - x}{2} + \dfrac{2 - x}{5} + \dfrac{1 - x}{10} = 0$.
(ii) Solve for f the equation $(u - f)(v - f) = (1 - f)^2$.

3. Simplify $\dfrac{1}{x + 3} - \dfrac{2(2 - x)}{3 - x} - \dfrac{x - 15}{x^2 - 9}$.

4. Factorise $(x - 1)(2x^2 + 5x + 3) + (x + 1)(2x^2 - 3x + 1)$.
For what values of x does the expression vanish.

5. The product of the three consecutive integers after x exceeds the product of the three consecutive integers before x by 600. Find x.

Test Paper 83.

1. An expression is divisible by $x^2 + 1$, and the quotient is divisible by $x^2 - x + 1$. If the final quotient is $x + 2$, find the expression.

2. Solve $\dfrac{4x - 7}{5} - \dfrac{15y + 7}{8} = \dfrac{2x - y - 1}{3}$

$$\frac{x - y}{4} = \frac{2x}{3} + 2y + 1.$$

3. (i) Factorise $x^2 + 4xy + 4y^2 - 4x - 8y + 4$.

(ii) Simplify $\left(\dfrac{a}{x} - \dfrac{x}{a}\right)\left(\dfrac{a}{x} + \dfrac{x}{a}\right)\left(1 - \dfrac{x - a}{x + a}\right)$.

4. Solve $\dfrac{x}{x - 1} - \dfrac{2x - 1}{x^2 + 2x - 3} = 1$.

5. A father's age is four times that of his eldest son and five times that of his youngest. When the eldest son has lived to three times his present age, the father's age will then exceed the combined ages of his sons by two years. Find their present ages.

Test Paper 84.

1. From a height h ft. above the earth, the fraction of the earth's surface which is visible is $\dfrac{h}{2h + d}$ where d is the earth's diameter in feet. Find the value of this fraction when $h = \dfrac{d}{n - 2}$.

2. Solve $3x - 5ay + 12 = 0$

$$2x + 3ay = 11.$$

3. Simplify $\left(\dfrac{ps + qc}{pc - qs} - \dfrac{s}{c}\right) \div \left(1 + \dfrac{ps + qc}{pc - qs} \times \dfrac{s}{c}\right)$.

4. Solve $(x + 1)^2 + (x + 2)^2 = (x + 3)^2 - (x - 2)^2$.

5. A man walks a certain distance in 6 hours. If he were to reduce his rate by one-seventh he would walk 3 miles less in the time. At what rate is he walking.

Test Paper 85.

1. If $a = 4{\cdot}5$, $b = 3{\cdot}2$, and $x = {\cdot}52$, evaluate

(i) $a^2 + b^2 - 2abx$.

(ii) $\dfrac{a - b}{a + b} \div \dfrac{a - b}{a + bx}$.

(iii) Write down two values of x which make (i) a perfect square.

2. Solve $lx - my = m - l$

$$x + y = l + m - 2.$$

3. Factorise (i) $a^3(3a - 4b) + 4ab(a^2 - 6b^2)$.
(ii) $(b - c)x^2 + (a + b - c)x + a$.

4. Solve (i) $20x^2 - 53x - 39 = 0$.
(ii) $2x^2 - 11x + 6 = 0$ (correct to two dec. places).

5. A man rows a mile up-stream and returns, taking altogether 32 min. If the speed of the current be 1 mile per hr., find his rate in still water, and the time taken going up-stream.

Test Paper 86.

1. A piece of work can be done by m men in x days and b boys can do the same work in y days. If a man receives 10/- a day, find the cost of doing the work, and how much per day a boy should be paid.

2. Solve (i) $\frac{7}{x} - 4y = 2$

$$\frac{4}{x} + 7y = 29.$$

(ii) $(2x - 19)(5x + 7) = 16x + 10$.

3. Simplify $\frac{a^2 - (b - c)^2}{(a + b)^2 - c^2} + \frac{b^2 - (c - a)^2}{(b + c)^2 - a^2} + \frac{c^2 - (a - b)^2}{(c + a)^2 - b^2}$.

4. Solve $\frac{2x - 3}{x - 4} - \frac{x - 1}{x - 2} = 1$.

5. In a cricket match between two sides A and B, B leads on the first innings by 95 runs. In the second innings A scores twice as many as B did in the first, and B scores three times as many as A did in the first. If A win by 3 runs, find the score of each side in each innings.

Test Paper 87.

1. On dividing a certain expression by $x + 1$ the remainder is 1. On dividing the quotient by $x + 1$ the final quotient is $x - 2$ and the remainder is 1. Find the original expression.

2. Solve $\frac{x^2 - a^2}{x - a} + \frac{x^3 - b^3}{x - b} - \frac{x^3 + c^3}{x + c} = a - b + c$.

3. Simplify $\frac{(x + y - z)^2 - w^2}{(x + y)^2 - (z + w)^2} + \frac{(y + z - x)^2 - w^2}{(y + z)^2 - (x + w)^2}$.

4. (i) Solve $\frac{1}{3}(x - 1) - \frac{1}{5}(y + 2) = 0$

$$\frac{1}{5}(x - 2) - \frac{1}{3}(y + 1) = 0.$$

(ii) Solve $(4x + 5)(5x - 4) = 78$.

5. A boy runs a mile race in $6\frac{1}{4}$ min. He does the last lap—one quarter mile—at 3 m.p.h. faster than he does the first three laps. Find his pace for the last lap.

Test Paper 88.

1. If $a = \dfrac{\mathrm{L} - l}{\mathrm{L}(\mathrm{T} - t)}$ change the subject to **L.**
2. Find the L.C.M. of
$$x^8 - 27x^5$$
$$x^2 + 5x - 24$$
$$(x + 2)^2 - (x - 8)^2.$$
3. (i) Solve $(a + b)x - (a - b)y = 1$
$(a - b)x + (a + b)y = 1.$
(ii) Solve $3ax^2 - x = \dfrac{24}{a}$.
4. If $x = \dfrac{a + 1}{ab + 1}$ and $y = \dfrac{a(b + 1)}{ab + 1}$, find the value of $\dfrac{x + y - 1}{x - y + 1}$.
5. A and B had £520 between them. A gains from B one-third of B's money, then B gains from A one-fourth of A's original money. They have now the same amounts. How much had each at first?

Test Paper 89.

1. If $x = \dfrac{2 - 3y}{5}$ and $z = \dfrac{5x - 8}{2}$, express z in terms of y.
2. Factorise (i) $3(x^3 + y^3) - xy(1 + 9xy)$.
(ii) $2a^2 - 3ab + b^2 + ac - bc$.
3. Simplify $\left(\dfrac{2}{x} - \dfrac{1}{x - y} - \dfrac{1}{x + y}\right) \div \left(\dfrac{x - y}{x + y} - \dfrac{x + y}{x - y}\right)$.
4. Solve $(2a - bx)(2b + ax) = (a + bx)(b - ax)$.
5. A carpet is laid on the floor of a room so as to leave a margin 1 ft. wide all round. If the breadth of the carpet is four-fifths of the length, and if the area of the margin is half the area of the carpet, find the dimensions of the latter.

Test Paper 90.

1. If $\dfrac{(p + 1)(q + 1)}{n + 2} = p + q$, prove $n + 1 = \dfrac{pq + 1}{p + q}$.
2. Simplify $\dfrac{\dfrac{1}{x} + \dfrac{1}{y}}{\dfrac{1}{x} - \dfrac{1}{y}} \div \dfrac{x^2 - xy - 2y^2}{x^2 + xy - 2y^2}$.
3. Factorise (i) $4a^2x + c^2y - 4a^2y - c^2x$.
(ii) $cx^3 + 2ay^3 - cy^3 - 2ax^3$.
4. Solve $\dfrac{6}{2x + 3} - 2 = \dfrac{3x - 7}{x + 7}$.

5. Write down expressions for three consecutive even numbers, the middle one being $2a$.

If x, y, and z are three consecutive even numbers prove

$$x^2 - 2y^2 + z^2 = 8.$$

Test Paper 91.

1. A man cycles a distance of $2a$ miles, the first half of the way at $m + 1$ miles per hour and the second half at $m - 1$ miles per hour. Show that if he had cycled steadily at m miles per hour he would have been $\frac{2a}{m(m^2 - 1)}$ hours less on the journey.

2. If $y + z = 2$, $z + x = 3$, and $x + y = 1$, find x, y and z.

3. Factorise (i) $(bc + ca + ab)^2 - b^2c^2 - c^2a^2 - a^2b^2$.

(ii) $(a + 2b)a^3 - (2a + b)b^3$.

4. Solve $\frac{x - 3}{x + 5} = \frac{(2x - 3)^2}{(2x + 5)^2}$.

5. A manufacturer finds that as he increases the price of a certain article the number sold per week decreases as follows :—

Price	2s.	2s. 6d.	3s.	4s.	4s. 6d.
Number Sold	18,000	14,600	12,000	5200	2800

The cost of production of each article is 1s. 6d. Draw a graph showing the total profit per week which he makes at the various prices.

Find the price which will give the maximum profit per week. Find also at what increased price he will probably obtain the same profit as when he sold the article at 2s.

Test Paper 92.

1. If $\frac{1 + x}{1 - x} = \frac{b}{a} \times \frac{1 + x + x^2}{1 - x + x^2}$ prove $x^3 = \frac{b - a}{b + a}$.

2. Find a value of m for which the following equations are satisfied by the same value of x.

$$\text{(i) } 8 - \frac{x - 2}{3} = 7.$$

$$\text{(ii) } (m - 1)x^2 - (3m - 4)x - (8m + 1) = 0.$$

Show that for this value of m there is another value of x which will satisfy the second equation, but not the first, and find the value.

3. If $x = \frac{2ab}{a + b}$ prove $\frac{1}{x - a} + \frac{1}{x - b} = \frac{2}{x}$.

4. There were 350 tickets sold for a concert. They cost 1s. and 2s. 6d., and the total amount drawn for shilling tickets was the same as for half-crown tickets. Find the number of each kind.

Prove that if $4n$ half-crown tickets and $5n$ shilling tickets are sold, the total amount drawn for the former is double that drawn for the latter.

5. Draw the graph of $2x^2 - 2x$ from $x = -1{\cdot}5$ to $x = 2{\cdot}5$. Read off two values of x that make $2x^2 - 2x = 3$, and also the minimum value of the expression.

Test Paper 93.

1. A train is scheduled to run a journey of D miles in h hr. Its average speed for the first d miles is m miles an hour. For the rest of the journey it travels at x miles an hour, and thereby makes up for lost time. Show that $x = \dfrac{m(\mathrm{D} - d)}{mh - d}$.

2. Solve
$$y + z = 2$$
$$z + x = 3$$
$$3x + 4y + 5z = 6.$$

3. Simplify $\dfrac{\dfrac{2}{a+b} - \dfrac{1}{a}}{b + \dfrac{ab}{b-2a}} + \dfrac{\dfrac{2}{a+b} - \dfrac{1}{b}}{a + \dfrac{ab}{a-2b}}$.

4. If a certain king had died 9 years younger, he would have reigned for the eighth part of his life, but if he had lived 9 years longer he would have reigned for the half of his life. At what age did he die, and how long did he reign?

5. Draw the graph of $4y = x^3 - 3x + 2$ from $x = -3$ to $x = 3$. If $x^3 - 3x = 6$, find from the equation the value of y, and from the graph the corresponding value of x.

Test Paper 94.

1. If $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{1}{5}$ show that $(u - 5)(v - 5) = 25$.

2. Solve $\dfrac{x}{a+b} + \dfrac{y}{a-b} = 2a$
$$\frac{x-y}{2ab} = \frac{x+y}{a^2+b^2}.$$

3. Factorise (i) $a^4b^2(a-b)^2 - a^2b^4(a+b)^2$.
(ii) $6x^3 + x^2 - 5x - 2$ given one factor to be $x - 1$.

4. A and B play two games at cards. In the first game A loses to B as much money as B has. In the second game B loses to A as much money as A now has. Each has now ten shillings. How much had each at first?

5. Draw the graph of $y = x^2 + x - 2$ from $x = -3$ to $x = 2$. Find the minimum value of y. Use the graph to solve the equation $x^2 + x = 3$.

Test Paper 95.

1. On AB and BC, sides of a square ABCD, points E and F are respectively taken so that AE = x and CF = $2x$. If EF = a, a side of the square, prove $x = \frac{1}{5}a$.

2. Show that one root of the equation $3(3x^2 + 6) = x(x + 25)$ also satisfies the equation $x^3 - 19x + 30 = 0$.
Find the other roots of the equations.

3. Simplify $\dfrac{3xyz}{xy + yz - xz} - \dfrac{\dfrac{x-1}{x} + \dfrac{y+1}{y} + \dfrac{z-1}{z}}{\dfrac{1}{x} - \dfrac{1}{y} + \dfrac{1}{z}}$.

4. In an examination 11 out of every 13 candidates passed ; but of the failures 18 were due to candidates leaving without handing in a paper. Had 9 of these stayed and succeeded in passing then 10 out of every 11 candidates would have passed. How many candidates were there ?

5. Draw the graph of $y = x^2(4 - x)$ from $x = 0$ to $x = 4$ and find the greatest value of y in the interval. Use the graph to find two values of x satisfying the equation $x^2(4 - x) = 6$.

Test Paper 96.

1. If $P = \frac{m}{2}(u + v)^2 + \frac{m}{2}(u - v)^2$ and $Q = \frac{m}{2}(u + v)^2 - \frac{m}{2}(u - v)^2$.

Simplify the right-hand side of each formula.

If $u = vx$ express $\dfrac{P}{Q}$ in terms of x.

2. Solve $y + \dfrac{1}{y} = \dfrac{26}{5}$. If $y = \dfrac{2x + 3}{x + 4}$ find x.

3. Factorise (i) $16x^4 - (x + 1)^4$; (ii) $(a + bx)^2 + x(a - b)^2$.

4. A certain number of two digits has its tens' digit greater by 3 than its units' digit. When the number is multiplied by the units' digit the product is 425. What is the number ?

5. F is a point on the hypotenuse AC of a right-angled isosceles triangle ABC. FD is perpendicular to BC and FE is perpendicular to AB. If AB = 7 ins. and BD = x ins., show that the area of the rectangle BDFE = $(7x - x^2)$ sq. in.

Now draw the graph of $7x - x^2$ and so find for what position of F the area of the rectangle is greatest.

Test Paper 97.

1. A number of two digits ends in 5. Prove that its square is obtained by adding to 25, one hundred times the product of the tens digit and the next higher digit.

2. Simplify $\dfrac{(x+y)^4 - (x^2-y^2)^2}{x^3 + 3x^2y + 3xy^2 + y^3}$.

3. Solve $\begin{aligned} 2x + 3y + z &= 0 \\ x + y + z &= 0 \\ 5x + 7y - 3z &= 12. \end{aligned}$

4. Solve $\dfrac{3x+7}{4} + \dfrac{4x-7}{7} = \dfrac{x+3}{x-6}$.

5. 100 sq. in. of tin is used to make a lidless box whose base is a square of side x inches. Its volume is given by the formula

$$V = \frac{x(100 - x^2)}{4}.$$

Plot V against x on squared paper, from $x = 0$ to $x = 10$ and so determine the girth of the biggest box that can be made.

Test Paper 98.

1. Prove that the sum of the squares of any three consecutive odd numbers increased by 1 is divisible by 12.

2. Solve $\dfrac{6x+3y}{4} - \dfrac{3y-2x}{6} = 7\frac{1}{4} - \dfrac{x+y}{3}$.

$$3x + 11y = 61.$$

3. Simplify (i) $\dfrac{x+2y}{x^2+xy-6y^2} - \dfrac{x-y}{x^2+7xy+12y^2} + \dfrac{x-4y}{x^2+2xy-8y^2}$.

(ii) $\dfrac{\dfrac{x}{c} + \dfrac{c}{x} - 2}{x-c} \cdot \dfrac{\dfrac{x}{c} + \dfrac{c}{x} + 2}{x+c}$.

4. Two towns A and B are 65 miles apart. A non-stop train running at 40 m.p.h. starts from A, and half an hour later a non-stop train running at 35 m.p.h. starts from B. How far from A will the trains pass each other?

5. Draw the graph of $y = \dfrac{3x}{x+6}$ from $x = -5$ to $x = 5$.

By drawing also the graph of $y = x - 3$ solve the equation

$$\frac{3x}{x+6} = x - 3.$$

Test Paper 99.

1. The sum of the length and girth of a package, of square section, is 6 ft. If the side of the section is x ft., express the length in terms of x, and prove that the volume of the package is $(6x^2 - 4x^3)$ c. ft.

2. Solve
$$3x + 4y + 5z = 31$$
$$4x + 3y + 2z = 32$$
$$5x + 6y - 9z = 33.$$

3. Solve $\dfrac{x-5}{x+2} = \dfrac{3x-3}{2x+1}$ (correct to two significant figures).

4. When a bullet is fired from a rifle, its height h ft. above the ground, at any point of its path, is given by the formula
$$h = \mathrm{K}t_1t_2,$$
where t_1 sec. is the time taken by the bullet to reach the point and t_2 sec. is the further time that elapses before it reaches the ground. If it is 24 ft. above the ground 1 sec. after being fired, and it takes $1\frac{1}{2}$ sec. more to reach the ground, find K, and its height after 2 sec.

5. Draw the graphs of $y = \dfrac{1}{x}$ and $y = x^2 + 2$ on the same diagram.
Hence find a value of x satisfying the equation $x^3 + 2x = 1$.

Test Paper 100.

1. Make a the subject of the following formula
$$\frac{\mathrm{PV}}{1 + a\mathrm{T}} = \frac{pv}{1 + at}.$$

2. Prove that
$$(2x + 1)(2x + 3)(2x + 5)(2x + 7) + 16 = (4x^2 + 16x + 11)^2.$$

3. Solve $\dfrac{x+6}{x+1} - \dfrac{5-x}{2x} = \dfrac{15}{8}$.

4. If the cost of 7 lb. of coffee be equal to that of 5 lb. of tea, and if 6 lb. more of coffee than of tea can be bought for £1 15/-, what is the price per lb. of the tea?

5. A rectangular beam is cut from a cylindrical log. If x is the fraction which the breadth of the beam is of the diameter of the log, then $y = x(1 - x^2)$ where y represents the comparative strength of the beam. Plot y against x from $x = 0{\cdot}1$ to $x = 0{\cdot}9$, and find what value of x gives the strength y, the greatest value possible.

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ANSWERS.

EXAMPLES I (p. 4).

1. (*a*) 7. (*b*) 4. (*c*) 3. (*d*) 12. (*e*) 14. (*f*) 3. (*g*) 9. (*h*) 0. (*i*) 14. (*j*) 20. (*k*) 2. (*l*) 1. (*m*) 14. (*n*) 13. (*o*) 2. (*p*) 13. (*q*) 17. (*r*) 7. (*s*) 10. (*t*) 0.

2. (*a*) 9. (*b*) 21. (*c*) 1. (*d*) 13. (*e*) 4. (*f*) 1. (*g*) 8. (*h*) 15. (*i*) 0. (*j*) 80. (*k*) 3. (*l*) 13. (*m*) 7. (*n*) 5. (*o*) 4.

3. (*a*) 10. (*b*) 6. (*c*) 18. (*d*) 0. (*e*) 24. (*f*) 2. (*g*) 0. (*h*) 3. (*i*) 3. (*j*) $10\frac{1}{2}$.

4. 4, 7, 10, 13, 16, 19.

5. 6.

6. 120 sq. ft.

7. 20 sq. in.

8. 120, 88, 56, 24.

9. 0, 1, $1\frac{1}{8}$, $1\frac{1}{2}$.

10. 1130.

11. 50, 40, $43\frac{1}{3}$, 50, 58.

12. $5\frac{1}{4}$.

13. 54 c. ft.

14. $x^2 = 4, 25, 49, 1$; $x^3 = 8, 125, 343, 1$.

15. (*a*) 9. (*b*) 64. (*c*) 1. (*d*) 54. (*e*) 48. (*f*) 4. (*g*) 25. (*h*) 15 (*i*) 63. (*j*) 48. (*k*) 108. (*l*) 144. (*m*) 12. (*n*) 4. (*o*) 12. (*p*) 37. (*q*) 20. (*r*) 8. (*s*) 5. (*t*) 3.

16. $2x = 0, 2, 4, 6, 8$; $x^2 = 0, 1, 4, 9, 16$; $3x = 0, 3, 6, 9, 12$; $x^3 = 0, 1, 8, 27, 64$.

17. 51, 36, 19, 0.

18. 40.

19. 1.

20. 44 miles.

21. $32\frac{1}{2}$.

22. $38\frac{1}{2}$ sq. in.

23. $13\frac{1}{2}$.

24. 156 ft., 24 ft.

25. 50·48 ft.

26. 1.

27. 62.

28. 1540 c. in.

29. 49.

30. 144.

31. 15 sq. ft.

32. 0, 14, 91, 285, 650, 1240.

33. £240.

34. $2\frac{5}{12}$.

35. 5, 17, 37 in.

EXAMPLES II (a) (p. 9).

1. 4. **2.** 5. **3.** 3. **4.** 6. **5.** 8. **6.** 7. **7.** 4. **8.** 6. **9.** 5. **10.** 7. **11.** 2. **12.** 9. **13.** 8. **14.** 24. **15.** 35. **16.** 6. **17.** 2. **18.** 3.

19. 7.	24. 3.	29. 3.	34. 4.	39. 18.	44. 8.
20. 9.	25. 4.	30. 7.	35. 5.	40. 7.	45. 2.
21. 3.	26. 2.	31. 3.	36. 4.	41. 10.	46. 1.
22. 2.	27. 4.	32. 3.	37. 20.	42. 4.	47. 1.
23. 4.	28. 5.	33. 1.	38. 6.	43. 3.	48. 2.
		49. 3.	50. 2.		

EXAMPLES II (b) (p. 11).

1. $5x$.	4. $6n$.	7. $2b$.	10. k.	13. $4y^2$.	16. $7a-7$.
2. $6a$.	5. $3b$.	8. $4c$.	11. $5t$.	14. pq.	17. $9x-3y$.
3. $6x$.	6. $5p$.	9. m.	12. $8x$.	15. $7x+5$.	18. $7b+3c$

19. $7x^2 + 5y^2$.
20. $10xy + 4ab$.
21. $4b$, bx, b^2.
22. $12x$/-.
23. $20p$ shillings, $3y$ ft., $8g$ pints.
24. $\frac{d}{12}$ shillings, $\frac{c}{20}$ tons, $\frac{m}{60}$ hrs.
25. 2.
26. 6.
27. $6x$ shillings.
28. $2l$ inches.
29. $3d$ pence, $\frac{d}{4}$ shillings.
30. $\frac{s}{16}$ shillings, $\frac{3s}{4}$ pence.
31. 2.
32. $(10x + 7)$ lb.
33. $(40s + 2)$ shillings.
34. $\frac{5x}{2}$, $\frac{ax}{2}$ pence.
35. £$\frac{x}{4}$.
36. $6x$ yd.

37. $6xy$, $20a^2$, $\frac{ab}{15}$, $\frac{3b}{ac}$, $15x$, $15x$, $24a$, $30b$, $18y$, $2c$, $\frac{m}{2}$, $\frac{3}{2z}$.

38. $2n + 1$.
39. $(10 + x)$ years, $(10 - x)$ years.
40. $44x$.
41. $\frac{3x}{4}$, $\frac{w}{2}$, $\frac{m}{12}$, $\frac{z}{12}$.

42. $\frac{3a + 2b}{6}$, $\frac{4b + 3c}{12}$, $\frac{3f - 2g}{12}$, $\frac{3g - 2h}{18}$.

43. $6x$ inches.

EXAMPLES III (a) (p. 14).

1. 5 ft. below sea level; £8 loss; £10 profit; 6 miles South; 7 miles East; A spring is compressed $1\frac{1}{2}$ in.; a fall in temperature of 14°; A man's debts are £60; A patient loses $2\frac{1}{2}$ lb. in weight; $3\frac{1}{4}$ gal. of water flow out of a tank; A patient's temperature rises 2°; The cost of living rises 6 points; An army advances 6 miles; A clock is 4 min. fast; The population of a town decreases by 2000.

2. A gain of −£7; A fall of −3°; −8 miles North; −£2 profit.
3. +34, −1, −39, −45, +76, +9, −77.
4. −5, +7, +11, −12 min.

5. + 10°, + 5°, 0°, − 5°, − 10°.
6. − 8°, − 4°, 0°, 6°, 15°.
7. − 22, + 10, + 29, − 17 per cent.
8. + 75, + 25, − 25, − 75, − 125 ft.
9. − 40, − 10, + 20, + 50 miles.

EXAMPLES III. (b) (p. 16).

1. 3.
2. − 5.
3. − 15.
4. $2x$.
5. $-3y$.
6. $-11y$.
7. $2w$.
8. $-4t$.
9. $-2m$.
10. $-22s$.
11. $3a$.
12. $10v$.
13. $-14x$.
14. 0.
15. $-4c$.
16. $4x^2$.
17. $-4xy$.
18. 0.
19. $5a^2$.
20. $-5xyz$.

EXAMPLES III (c) (p. 17).

1. 8.
2. − 10.
3. − 4.
4. − 11.
5. + 5.
6. 16.
7. $5x$.
8. $-5a$.
9. $-10c$.
10. $12c^2$.
11. $-11xy$.
12. $15pq$.
13. 15.
14. − 6.
15. 4.
16. $5a$.
17. $-5a$.
18. $-10x^2$.
19. $-5ab$.
20. $-10pq$.
21. $+11c^2$.
22. 78° F.
23. 35° C.
24. £300.

EXAMPLES III (d) (p. 19).

1. 3.
2. − 3.
3. 2.
4. − 2.
5. 4.
6. − 4.
7. − 4.
8. − 10.
9. 5.
10. − 13.
11. $4x$.
12. $-3a$.
13. 0.
14. 0.
15. $2w$.
16. $-4a$.
17. $-9a$.
18. $4p$.
19. $-9b$.
20. $-9c$.
21. $3x$.
22. $-7y$.
23. z.
24. $-5t$.
25. $-7s$.
26. $-3c$.
27. $-7c$.
28. $-3m$.
29. $-13q$.
30. 0.

EXAMPLES III (e) (p. 19).

1. $x + y$.
2. $-8a - 4b$.
3. $2r - 2t$.
4. $9s + 5w$.
5. $8b + 8c$.
6. $-6a + 4b$.
7. $5p + 6q$.
8. $13y + 10z$.
9. $6x - 10y$.
10. $-8s - 8t$.
11. $-3v^2 - 2w^2$.
12. $-3xy - ab$.
13. $5a - 5b + c$.
14. $-3x - 5y - z$.
15. $2x^2 - 14a^2$.
16. $2p^2 - 4q^2 + 3r^2$.
17. $-3x^2 + 3x + 3$.
18. $9y^2 + 5y - 7$.
19. $2t^2 - 7t + 7$.
20. $5a - 8b$.
21. $-2x + 13y$.
22. $8m$.
23. $-x^2 + 7y^2$.
24. $-3a^2 + 7xy$.

EXAMPLES III (f) (p. 20).

1. $-2a - 5b$.
2. $9a + 3b$.
3. $-3x^2 - 6y^2$.
4. $2x - 11y$.
5. $6c + 4d$.
6. $12a^2 - 5b^2$.
7. $-4x^2 - 6ab$.
8. $7x^2 + 6y^2$.
9. $-11pq - 8rs$.
10. $-10ab + 14c^2$.
11. $-5t^2 + 10$.
12. $16bc - 2a^2$

13. $5x^2 - 9x + 7$.
14. $7y^2 - 2y - 14$.
15. $6a - b - 15c$.
16. $-2p^2 + 7q^2$.
17. $4a - 12b - 2c$.
18. $6p - 2q - 9r$.
19. $-3x - 5y - 4z$.
20. $3x + 3y - 10z$.
21. $-3a^2 + 10b^2 + 7c^2$.
22. $-2y^2 + y - 7$.
23. $y^2 - 5y - 2$.
24. $5y^2 + 6$.
25. $x + 3$.
26. $-9a + 2b$.
27. $7x - 2y$.
28. $-9p + 10q$.
29. $4s - 8t$.
30. $-3a - 5b - 2c$; $-6b - 8c$.
31. $-3x - 8y + 9z$.
32. $-3x^2 + 5y^2 + z^2$

EXAMPLES IV (a) (p. 24).

1. -14, -30, -32.
2. -27, -56, -66.
3. 10; 24; 63.
4. -40; 27, -56.
5. 42; 24; -49.
6. -25; -64; 144.
7. -20; 72, 26.
8. -140.
9. -56.
10. -80.
11. 108.
12. 192.
13. 120.
14. -120.
15. 120.
16. -120.
17. -12.
18. 32.
19. 90.
20. -88.
21. -16.
22. 45.
23. 4, 4, 9, 9, 25, 25, 49, 49, $\frac{1}{16}$, $\frac{1}{36}$, $\frac{1}{64}$.
24. -8, -64, -216, -512, $-\frac{1}{27}$, $-\frac{1}{125}$, $-\frac{1}{343}$.
25. 48.
26. -48.
27. -4096.
28. 16; 81; 256; 625.
29. 100.
30. 13, 10, -8, -11.
31. 46, 21, -4, -19
32. 15, 10, 0, -2.

33.

x	-3	-2	-1	0	1	2	3
y	14	11	8	5	2	-1	-4

34.

t	-5	-3	-1	0	2	4
u	20	8	-4	-10	-22	-34

EXAMPLES IV (b) (p. 26).

1. -4; -2; 3.
2. -7; 6; -8.
3. 5; -4; -5.
4. -15; 9; -7.
5. -9; -14; 9.
6. 13; -9; -7.
7. -4.
8. -6.
9. 6.
10. 9.
11. $4\frac{1}{3}$.
12. -4.
13. -18.
14. 3.

EXAMPLES IV (c) (p. 28).

1. $10ab$; $21bc$; $12pq$; $54xy$; $24abc$; $20fgh$; $18pqr$; $18xyz$.
2. $24ab$; $-20xy$; $-27st$; $-40pqr$; $90lmn$; $20abc$; $-42cfg$.
3. x^5 ; x^8 ; x^7.
4. a^7 ; a^{10} ; a^{12}.
5. v^5 ; v^{10} ; v^6.
6. y^{10} ; y^{12} ; y^{11}.
7. $-x^8$; x^{10} ; $-x^{10}$.
8. m^{10} ; m^{12}.
9. z^{16}.
10. $6x^3$; $20x^5$.
11. $21y^5$; $30y^6$.
12. $-32b^5$; $30b^{10}$.
13. $18s^9$; $-28s^9$.
14. $-60f^{15}$.
15. $12a^5b$; $10a^4b^3$.
16. $14x^7y^7$; $36x^7y^{13}$.
17. $-40a^3b$.
18. $30a^3b^2$.
19. $-24a^2b^2$.
20. $-64a^2c^2d^2$.
21. $-18a^5b^4$.
22. x^3 ; x^4 ; x^8.
23. $-y^4$; $-y^5$; $-y^9$.
24. $-z^3$; $-z^4$; $-z^8$.
25. a^4 ; a^8 ; a^6.
26. $2t^4$; $2t$; $5t^4$.
27. $-3s^5$; $-2s^6$; $2s^6$.
28. a ; a^2b^2 ; $3ab^4$.
29. $-3b^4c^5$; $4b^2c^3$; $-9bc^8$.
30. $-2b$; $5b$; $-6c^3$.
31. $\frac{1}{b^3}$; $\frac{b^3}{4a^2}$; $\frac{3}{a^2b^3}$.

EXAMPLES V (a) (p. 30).

1. $2x + 2$.
2. $5b - 10$.
3. $-4v - 8$.
4. $-6m + 30$.
5. $8 + 2a$.
6. $15 - 3a^2$.
7. $-24 - 4c$.
8. $7t^3 - 14$.
9. $8st + 24$.
10. $4x + 4y$.
11. $-5y - 20z$.
12. $-12a^2 + 18b^2$.
13. $3p - 6q - 12r$.
14. $-18f + 27g + 9h$.
15. $ay + az$.
16. $bm + 2bn$.
17. $3dr - ds$.
18. $-mx - 5my$.
19. $-px - 3py$.
20. $-ks - 2kt$.
21. $2ax + 4bx$.
22. $6at - 15bt$.
23. $-12su + 20sv - 8sw$.
24. $x^2 + xy$.
25. $xy - y^2$.
26. $a^3 + a^2b$.
27. $-a^2b^3 + b^5$.
28. $-2w^3 + 2w^2x$.
29. $p^3q - pq^3$.
30. $-3m^4s + 6ms^4$.
31. $2y^3z - 3y^2z^2 - 4yz^3$.
32. $3a^3b - 12a^2b^2 + 15ab^3$.
33. $-6b^3c - 4b^2c^2 + 14bc^3$.

EXAMPLES V (b) (p. 31).

1. $7x + 4$.
2. $13y - 30$.
3. $3pz - 30$.
4. $24t^2 + 4$.
5. 6.
6. $12x$.
7. $18a - 24b$.
8. $2b - 16c$.
9. $5y + 15z$.
10. $36p - 3q$.
11. $3t - 9u$.
12. $14a - 10b$.
13. $21b - 28c$.
14. $2m - 8n$.
15. $3b^3 + 4c^3$.
16. $-20p - 11q$.
17. $14a - 9b$.
18. $-6m^2 + 17n^2$.
19. $x^2 + xy - 4y^2$.
20. $s^4 - 3s^2t^2 - 4t^4$.
21. $a^2 - ac + bc$.
22. 0.
23. $a^2 + b^2 + c^2$.
24. $10a$.
25. $-12y$.
26. $12x^2 - 9y^2$.
27. $-7m + n$.
28. $-13y - 23z$.
29. $3a^2 + 5b^2$.
30. $a^2 - ab - bc + c^2$.

EXAMPLES VI (a). (p. 33).

1. $x^2 + 5x + 6$.
2. $x^2 + x - 12$.
3. $x^2 - 17x + 72$.
4. $y^2 - y - 42$.
5. $30 + 7p - p^2$.
6. $2x^2 + 3x + 1$.
7. $3x^2 - 14x + 8$.
8. $6x^2 - x - 12$.
9. $8m^2 - 18$.
10. $28t^2 + 9t - 4$.
11. $3a^2 + 10ab + 3b^2$.
12. $a^2 - 16b^2$.
13. $9m^2 - 6mn + n^2$.
14. $8 - 6a - 5a^2$.
15. $6 + 8a - 30a^2$.
16. $10y^2 - 54yz + 56z^2$.
17. $x^4 - 6x^2y^2 + 5y^4$.
18. $2a^4 - 7a^2b^2 + 6b^4$.
19. $x^3 + 2x^2 + 5x + 4$.
20. $x^3 - 5x^2 - 11x + 6$.
21. $x^3 - 6x^2 - 30x - 9$.
22. $x^3 + 6x^2 - 22x + 12$.
23. $a^3 - 10a^2 + 19a + 20$.
24. $-y^3 + 12y^2 - 32y - 15$.
25. $3x^3 + 10x^2 - 10x - 24$.
26. $2b^3 - 9b^2 + 19b - 15$.
27. $-4c^3 + 22c^2 - 2c - 4$.
28. $4w^3 - 8w^2 - 27w - 9$.
29. $6x^3 + 4x^2 - 34x + 24$.
30. $10m^3 - 25m^2 + 21m - 9$.
31. $-9r^3 - 9r^2 + 58r - 30$.
32. $12y^3 - 29y^2 + 11y + 3$.
33. $1 - 2m - 9m^2 + 4m^3$.
34. $12 - 10a + 12a^2 + 10a^3$.
35. $x^3 + y^3$.
36. $m^3 - 27n^3$.
37. $6y^3 + 4y^2z - 34yz^2 + 24z^3$.
38. $-6a^3 + 11a^2b + 7ab^2 - 15b^3$.
39. $12b^3 - 16b^2c - 13bc^2 + 2c^3$.
40. $4p^4 - 18p^3 + 16p^2 + 16p - 15$.
41. $6x^4 + 10x^3 + 10x^2 + 18x - 20$.
42. $16x^4 + 12x^3 - 14x^2 - 22x + 12$.
43. $8 - 26a + 37a^2 - 32a^3 + 7a^4$.
44. $x^4 + 3x^3 + 6x^2 + 5x + 3$.
45. $2a^4 - 3a^3 + 6a^2 - a - 12$.
46. $b^4 + 4$.
47. $4y^4 + 4y^3 - 15y^2 + 2y + 5$.
48. $-3c^4 + 10c^3 + 6c^2 - 20c + 8$.
49. $8z^4 - 38z^2 - 18z + 8$.
50. $3x^4 - 7x^3 + 13x^2 + 3x - 20$.
51. $x^4 + 2x^2y^2 + 9y^4$.
52. $2y^4 + 7y^3z - 7y^2z^2 + 6yz^3 - 2z^4$.
53. $6a^4 - 13a^3b + 7a^2b^2 + 6ab^3 - 8b^4$.
54. $12b^4 - 22b^3c - 15b^2c^2 + 24bc^2 + 10c^4$
55. $12 + 5a - 9a^2 - a^3 + a^4$.
56. $x^5 - x^4 - 2x^3 - x^2 - 9x - 4$.
57. $4x^2 - 9y^2 + 6yz - z^2$.
58. $a^3 + b^3 - c^3 + 3abc$.
59. $5x^6 - 18x^4 + 8x^3 - 8x^2 + 34x - 21$.
60. $8a^4 - 2a^3b + 5a^2b^2 + 9ab^3 - 6b^4$.

EXAMPLES VI (b). (p. 35).

1. $x + 4$.
2. $y + 3$.
3. $a - 9$.
4. $b + 8$.
5. $p - 7$.
6. $2x + 5$.
7. $3x - 2$.
8. $2y - 4$.
9. $3m + 2$.
10. $2c - 3$.
11. $2b + 3$.
12. $3m - 7$.
13. $5p - 3$.
14. $x - 8$. Rem. 18.
15. $a + 8$. Rem. 11.
16. $x - 11$. Rem. 17.
17. $p + 5$. Rem. 5.
18. $3c + 2$.
19. $5 - 2p$.
20. $6 + w$.
21. $x^2 - 5$. Rem. 1.
22. $m + n$.
23. $2a + 3b$.
24. $3s + 8t$.
25. $a - 4b$. Rem. b^2.
26. $x^2 + 2x + 1$.
27. $x^2 + x + 3$.

28. $b^2 - 3b - 4$.
29. $x^2 - x - 9$. Rem. -6.
30. $t^2 - 2t + 4$.
31. $y^2 + 3y + 9$.
32. $2z^2 + 3z + 4$.
33. $3r^2 + r - 4$.
34. $6a^2 - 2a - 3$.
35. $x^3 + x^2y + xy^2 + y^3$.
36. $x - 5$.
37. $2x - 5$.
38. $4m + 1$.
39. $3m + 2$.
40. $3m + 5$.
41. $5m - 7$.
42. $2b - 9c$.
43. $x + 3$. Rem. -1.
44. $a + 7$. Rem. $a - 45$.
45. $y - 4$. Rem. $-14y - 29$.
46. $2z - 4$.
47. $x^2 + x + 1$.
48. $x^2 + x + 1$.
49. $y^2 - 4y + 3$.
50. $a^2 + 3a - 2$.
51. $8 + 4c$.
52. $2x^2 + x - 3$.
53. $2x - 1$. Rem. $2x + 6$.
54. $x^2 + 4x + 15$. Rem. $6x - 5$.
55. 6.
56. $2x^3 + 9x^2 - 9x - 2$.

EXAMPLES VII (a) (p. 39).

1. 4.
2. 3.
3. $3\frac{1}{2}$.
4. $5\frac{3}{5}$.
5. 0.
6. -6.
7. -7.
8. -4.
9. $-3\frac{1}{3}$.
10. $-2\frac{1}{3}$.
11. -3.
12. -8.
13. 0.
14. 24.
15. 2.
16. 4.
17. 4.
18. 56.
19. $\frac{a}{3}$.
20. $2b$.
21. $-30a$.
22. $\frac{9}{v}$.
23. $\frac{12}{y}$.
24. $\frac{v}{f}$.
25. 2.
26. 7.
27. 5.
28. 9.
29. 8.
30. 7.
31. 3.
32. -4.
33. -6.
34. -12.
35. -11.
36. 18.
37. 3.
38. 9.
39. -3.
40. -4.
41. $-5\frac{1}{2}$.
42. $-1\frac{1}{2}$.
43. -5.
44. -5.
45. 0.
46. $8\frac{1}{2}$.
47. 3.
48. $-1\frac{1}{2}$.
49. 0.
50. 5.
51. -4.
52. 1.
53. -2.
54. -5.
55. $\frac{4+a}{2}$.
56. $\frac{b-6}{3}$.
57. $\frac{c+8}{7}$.
58. $\frac{b+2}{a}$.
59. $\frac{b+4}{a}$.
60. $\frac{9+d}{c}$.

EXAMPLES VII (b) (p. 39).

1. 4.
2. 3.
3. -3.
4. -2.
5. 2.
6. -6.
7. -4.
8. $2\frac{1}{2}$.
9. 6.
10. $7\frac{1}{2}$.
11. $2\frac{2}{5}$.
12. -2.
13. 0.
14. 3.
15. $-2\frac{1}{8}$.
16. $1\frac{1}{2}$.
17. 0.
18. -16.
19. 6.
20. -6.
21. -7.
22. $1\frac{1}{7}$.
23. -5.
24. $5\frac{1}{3}$.
25. $2\frac{4}{5}$.
26. 3.
27. 29.
28. 10.
29. 0.
30. $-6\frac{1}{2}$.

EXAMPLES VII (c) (p. 41).

1. 6.
2. − 30.
3. 18.
4. − 18.
5. − 6.
6. 16.
7. 20.
8. $-7\frac{1}{2}$.
9. 4.
10. 4.
11. − 9.
12. − 15.
13. $4\frac{1}{2}$.
14. $8\frac{4}{5}$.
15. 1.
16. $-\frac{14}{15}$.
17. $-\frac{3}{8}$.
18. 28.
19. 10.
20. − 18.
21. 2.
22. − 7.
23. $-\frac{4}{3}$.
24. 4.
25. − 57.
26. $-3\frac{2}{3}$.
27. 10.
28. − 41.
29. 10.
30. 3.
31. 15.
32. $\frac{1}{4}$.
33. 3.
34. − 5.
35. − 4.
36. − 7.
37. − 7.
38. − 6.
39. 5.
40. − 2.
41. $-10\frac{1}{12}$.
42. 9.
43. − 4.
44. 25.
45. 5.
46. 4.
47. $-\frac{2}{3}$.
48. 2.
49. $-2\frac{2}{5}$.
50. 3.
51. 8.
52. − 7.
53. − 5.
54. 2.
55. 0.
56. − 2.
57. − 5.
58. $-2\frac{1}{4}$.
59. 4.
60. 3.

EXAMPLES VII (d) (p. 43).

1. 30.
2. 20.
3. 4.
4. 3.
5. 4.
6. − 3.
7. − 1·8.
8. − 6.
9. − 9.
10. 7.
11. ·7.
12. − 5·5.
13. 3.
14. 15·5
15. − 3.
16. $-7\frac{1}{3}$.
17. 5·6.
18. − 3.
19. 3.
20. − 6.
21. $22\frac{1}{2}$.
22. $\frac{16}{31}$.
23. 5.
24. 8
25. − 8·75.
26. 4.
27. $2\frac{9}{16}$.
28. − 16.
29. 25.
30. − 1.
31. − 4.
32. $\frac{2A}{b}$.
33. $2c - 18$.
34. $\frac{y - 6}{m}$.
35. $\frac{q - p}{w}$.

EXAMPLES VIII (a) (p. 48).

1. 5, 2.
2. 7, 2.
3. 2, 7.
4. 1, 2.
5. 4, 8.
6. 2, − 3.
7. 2, 1.
8. 4, − 2.
9. − 5, 3.
10. − 2, 4.
11. 1, − 1.
12. 6, 2.
13. 3, 2.
14. 2, − 1.
15. − 1, 4.
16. 4, 3.
17. 2, − 7.
18. 6, 8.
19. 2, − 7.
20. 2, 4.
21. 2, 7.
22. $-\frac{1}{2}$, $-\frac{1}{4}$.
23. 3, 2.
24. 1, 4.
25. − 2, 3.
26. $\frac{1}{2}$, 1.
27. 3, 5.
28. 2, − 4.
29. 4, − 6.
30. $\frac{2}{3}$, $-\frac{1}{2}$.
31. 111, 11.
32. 2, − 4.
33. − 5, − 3.
34. − 7, 3.
35. 1, 4.
36. 3, 6.
37. 4, − 2.
38. $\frac{1}{3}$, − 1.
39. $\frac{1}{5}$, $-\frac{1}{3}$.
40. $\frac{1}{4}$, $\frac{1}{3}$.
41. 1, 1.
42. 2, 2.
43. − 4, 26.
44. 8, 3.
45. 6, 8.
46. $\frac{8}{13}$, $-\frac{12}{13}$.
47. 48.
48. 7.
51. 6.
52. $2\frac{1}{2}$.
53. 4, 5.

EXAMPLES VIII (b) (p. 50).

1. 9, 4.
2. 4, 5.
3. 6, − 2.
4. 6, − 8.
5. 2, − 1.
6. $\frac{1}{2}$, $\frac{1}{3}$.
7. 5, $5\frac{1}{2}$.
8. 3, 12.
9. $-\frac{3}{4}$, 4.
10. − 7, − 17.
11. 5, − 2.
12. 2, 3.
13. 6, − 4.
14. − 1, 2.
15. $3\frac{3}{5}$, $-1\frac{1}{5}$.
16. − 5, 25.
17. 12, 6.
18. $-\frac{3}{4}$, $5\frac{1}{4}$.
19. 8, 17.
20. 3, − 3.
21. 1, 2.
22. 5, − 3.
23. 3, − 2.
24. 5, $-1\frac{1}{2}$.
25. − 4, − 6.
26. $9\frac{1}{5}$, $-\frac{3}{8}$.
27. 2, 3.
28. − 1.
29. $-5\frac{3}{5}$.
30. 2, 6, 3.

31. $-4, 2, -1.$ 32. $5, -2.$
35. $-6, 8.$ 36. $-3, -7.$ 37. $-3c, 9c.$
38. $c-1, 2c+2.$ 39. $3, -4, 2.$ 40. $c.$

EXAMPLES VIII (c) (p. 53).

1. 1, 1, 2. 3. $-3, \frac{1}{2}, 1.$ 5. $1, -3, 2.$ 7. 3, 9, 6.
2. 1, 2, 3. 4. $-1, 0, 2.$ 6. $5, 1\frac{3}{7}, 2.$ 8. 5, 6, 7.

EXAMPLES IX (p. 56).

1. Boy, 7 st. Sister, 5 st.
2. 3 quarts. 3. 4.
4. Man, 12 st. Boy, 4 st.
5. Father, 35 yr. Son, 5 yr.
6. Father, 37 yr. Son, 12 yr.
7. 3. 8. 12 lb ; 4 lb.
9. 2s. ; 8d. 10. 1 wk.
11. 12 in., 4 in. 12. 12.
13. Box, 6 dr. Matches, 12 dr.
14. 50°, 50°, 80°. 15. 6.
16. 12, 8. 17. 11, 5.
18. 15, 6. 19. 27, 3.
20. 42, 6. 21. 5, 4.
22. Lime juice, 2/-. $\left\{\begin{matrix}\text{Lemon}\\\text{squash, 3/-.}\end{matrix}\right.$
23. 17. 24. 10.
25. 25, 20. 26. 15s. ; 10s.
27. £10, £20, £15.
28. 78°, 86°, 94°, 102°.
29. 11, 12, 13. 30. 5, 6, 7.
31. 6 in., 12 in.
32. 2s. 6d. ; 1s. 6d. ; 1s.
33. 71°, 59°. 34. 3/-, 1/-.
35. 5/-, 7/-. 36. 80°, 40°, 60°.
37. AB = 4 in., BC = 10 in., CD = 16 in.
38. 57°, 59', 64°. 39. 3 miles.
40. 2s. 6d. ; 3s. ; 3s. 6d.
41. 50 ac. wheat ; 10 ac. barley ; 30 ac. oats.
42.
43. 5 ft. 2 in. ; 4 ft. 8 in. ; 4 ft. 2 in.
44. 20 in. 45. 6s. ; 4s.
46. Calf, £5. Pig, £3.
47. 40 qr. oats, 2 qr. barley.
48. Length, 12 in. Breadth 8 in.
49. Tea, 3s. Coffee, 2s.
50. £2 ; £7. 51. 13/4, 5/-.
52. 37, 26. 53. 68, 57.
54. A, £72. B, £28.
55. Man, 40 yr. Son, 12 yr.
56. 10 h. cr., 7 fl.
57. 10 h. cr., 15s.
58. 44 hr. ordinary, 12 hr. overtime.
59. "First Early," 3s. 6d. per stone. "Maincrop," 2s. per stone.
60. Apples, 10d. per lb. Pears, 1s. 2d. per lb.
61. Tea, 2s. 2d. per lb. Coffee, 1s. 8d. per lb.
62. 72 lb. at 3s. 4d., 40 lb. at 2s. 6d.
63. Tea, 3/- per lb. Sugar 6d. per lb.
64. A, £24. B, £18.
65. 10 in. long, 4 in. wide, 3 in. deep.
66. A, £36. B, £14.
67. 14. 68. £2 10/-. 69. 4 hits.
70. 20 m.p.h., 24 m.p.h.
71. 47. 72. 51. 73. 25.
74. Single, 2s. Return, 3s. 3d.
75. Fresh butter, 2s. 11d. Salt butter, 2s. 8d.
76. Income, £400.
77. 21, 24, 30.
78. 40 miles by train, 60 miles by motor.
79. A, £80. B, £90. C, £230.
80. 60. 81. P, £36. Q, £24.
82. £1930, £2050, £2020.

34
15
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EXAMPLES X (a) (p. 64).

1. (1) Feb., March, Nov. (2) Jan., Sept., Dec.; 1·1 in., 1·8 in., 1·3 in. (3) Dry. (4) Oct. (5) Spring. (6) (*a* March—2·4 in. (*b*) May—0·1 in.
2. 60·5°, 52·5°, 42·5°, 38°; 5 min., 12·5 min., ; beginning.
3. 110 min., $3\frac{1}{2}$ lb.
4. $6\frac{3}{8}$ lb.
5. £15 12/-, 18 records.
6. 2·45 c. ft., 4·45 in.
7. 1500 yd.
8. 1·8 min., 3·95 min., 2300 yd.
9. £24, £37.
10. 35·54 lb.

EXAMPLES X (b) (p. 71).

1. 13 ft. 6 in.
2. 28·6 lb., 12·3 kg., 18·6 kg.
3. 14 sec., 22·5 in.
4. 3.36 P.M.; 12 min. earlier.
5. 3.16 P.M., $39\frac{1}{4}$ min., 2.20 P.M., 4.9 P.M.
6. 3.40 P.M. and 5 P.M.; 8 miles and 12 miles from starting-place.
7. $3\frac{3}{4}$ miles.
8. Area 6·25 sq. ft., length of sides, 2·5 ft., 2·5 ft., 3·62 ft. by 1·38 ft.
9. 9·72 in. by 7·72 in.
10. 5d.; 1·5d. or 8·5d.
11. 13·86 in.
12. (i) AP = 2·85 in. or 5·15 in.; (ii) AP = 4 in.

EXAMPLES XI (a) (p. 74).

1. $x^2 + 3x + 2$.
2. $a^2 + 7a + 12$.
3. $x^2 + 11x + 18$.
4. $x^2 + 12x + 32$.
5. $b^2 - 6b + 8$.
6. $p^2 - 8p + 15$.
7. $p^2 - 2p - 15$.
8. $y^2 + y - 42$.
9. $x^2 + 2x - 24$.
10. $x^2 - 3x - 70$.
11. $y^2 - 18y + 72$.
12. $20 + 9y + y^2$.
13. $28 + 3p - p^2$.
14. $18 - 9x + x^2$.
15. $x^2 + 5xy + 4y^2$.
16. $x^2 - 7xy + 10y^2$.
17. $y^2 + 3yz - 28z^2$.
18. $2x^2 + 5x + 3$.
19. $2x^2 + x - 6$.
20. $2k^2 - 3k - 2$.
21. $3y^2 + 7y - 6$.
22. $4a^2 + 16a + 15$.
23. $3y^2 - 10y + 7$.
24. $5x^2 - 29x - 6$.
25. $6m^2 + 13m + 6$.
26. $5x^2 - 22x + 8$.
27. $8a^2 - 6a + 1$.
28. $21x^2 + 11x - 2$.
29. $6y^2 - 13y - 8$.
30. $24x^2 - 14x - 3$.
31. $4m^2 + 8mn - 45n^2$.
32. $15y^2 - 11xy + 2x^2$.
33. $9x^2 - 18xy - 16y^2$.
34. $16x^2 - 8ax - 15a^2$.
35. $28x^2 + 9xy - 9y^2$.
36. $21y^2 - 5xy - 6x^2$.
37. $10x^2 - 29xy + 10y^2$.
38. $6p^2q^2 + 11xy - 10y^2$.
39. $2a^2 - 11abc + 5b^2c^2$.
40. $2x^2 + 12x + 5$.
41. -12.
42. $a^2 + a$.
43. $p^2 - 10p$.
44. $c^2 + 1$.
45. 3.
46. $4x^2 - 13x + 25$.
47. $-3x - 4$.
48. $x^2 + 10ax - 7a^2$.

49. 9. 50. 23. 51. $5\frac{1}{2}$. 52. 8. 53. 1. 54. − 1.
55. $6\frac{1}{2}$. 56. 1. 57. 10. 58. 6. 59. $1\frac{2}{3}$.

EXAMPLES XI (b) (p. 76).

1. $x^2 + 8x + 16$.
2. $y^2 + 2y + 1$.
3. $m^2 + 14m + 49$.
4. $p^2 - 10p + 25$.
5. $9 - 6m + m^2$.
6. $a^2 - 20a + 100$.
7. $x^2 + 4xy + 4y^2$.
8. $x^2 - 8xy + 16y^2$.
9. $1 - 6a + 9a^2$.
10. $1 - 16b + 64b^2$.
11. $9a^2 + 12a + 4$.
12. $4l^2 - 12l + 9$.
13. $16m^2 - 40m + 25$.
14. $4x^2 + 28x + 49$.
15. $25p^2 - 40pq + 16q^2$.
16. $49x^2 - 28xy + 4y^2$.
17. $9a^2 + 24ab + 16b^2$.
18. $49p^2 - 70pq + 25q^2$.
19. $100a^2 + 20ab + b^2$.
20. $4x^2 - 20xy + 25y^2$.
21. $x^4 - 8x^2y^2 + 16y^4$.
22. $4m^4 - 36m^2n^2 + 81n^4$.
23. $a^2b^2 + 4abc + 4c^2$.
24. $9x^2 - 12xyz + 4y^2z^2$.

25. (i) $a + b$. (ii) $a - b$. (iii) $x - 3y$. (iv) $p + 2q$. (v) $m - 5$. (vi) $2c - 3d$. (vii) $3a - 2b$. (viii) $5a - 2$. (ix) $4x + 3y$.

26. $a^2 - 9$.
27. $c^2 - 25$.
28. $p^2 - 1$.
29. $49p^2 - 100$.
30. $16m^2 - 9n^2$.
31. $a^2 - 64b^2$.
32. $4p^2 - 9q^2$.
33. $25m^2 - 81n^2$.
34. $100a^2 - 9b^2$.
35. $25x^2 - 144y^2$.
36. $1 - 121x^4$.
37. $a^4 - 4b^2$.
38. $36 - x^4$.
39. $16x^2 - 9a^4$.
40. $25p^2q^2 - 121r^2$.
41. $9a^2$.
42. $25p^2 - 22pq - 5q^2$.
43. $4xy$.
44. a^2.
45. $3m^2 - 20mn + 38n^2$.
46. − 1.

47. 2. 48. − 1. 49. 6. 50. $2\frac{6}{7}$.

EXAMPLES XI (c) (p. 77).

1. $a^2 + 4b^2 + c^2 + 4ab + 2ac + 4bc$.
2. $x^2 + y^2 + z^2 + 2xy - 2xz - 2yz$.
3. $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$.
4. $4x^2 + 9y^2 + z^2 + 12xy + 4xz + 6yz$.
5. $4x^2 + y^2 + 9z^2 - 4xy + 12xz - 6yz$.
6. $9x^2 + 4y^2 + a^2 - 12xy + 6ax - 4ay$.
7. $16a^2 + b^2 + 4c^2 + 8ab - 16ac - 4bc$.
8. $25x^2 + 36y^2 + z^2 - 60xy + 10xz - 12yz$.
9. $9x^2 + 16y^2 + z^2 + 24xy - 6xz - 8yz$.
10. $4l^2 + 25m^2 + 4n^2 + 20lm + 8ln + 20mn$.
11. $9p^2 + q^2 + 4r^2 - 6pq - 12pr + 4qr$.
12. $p^2 + 16q^2 + 9r^2 + 8pq - 6pr - 24qr$.
13. $a^2 + 2ab + b^2 - 9$.
14. $a^2 - 2ab + b^2 - 25$.
15. $a^2 - 4b^2 - 4bc - c^2$.
16. $a^2 - 9b^2 - 12bc - 4c^2$.
17. $4x^2 + 4xy + y^2 - 36z^2$.
18. $9x^2 - y^2 + 4yz - 4z^2$.

EXAMPLES XII (p. 80).

1. $(x+1)(x+2)$.
2. $(x+1)(x+3)$.
3. $(x+2)(x+3)$.
4. $(x-2)(x-4)$.
5. $(y-2)(y-5)$.
6. $(x+2)(x+9)$.
7. $(a-1)(a-7)$.
8. $(b-1)(b-4)$.
9. $(x+4)(x-1)$.
10. $(y+4)(y-3)$.
11. $(x-7)(x+6)$.
12. $(x-1)(x-10)$.
13. $(m+7)(m+8)$.
14. $(a+9)(a-8)$.
15. $(x+7)(x-2)$.
16. $x(x-8)$.
17. $x(x+12)$.
18. $a(a-18)$.
19. $(a+2)^2$.
20. $(y-5)(y+3)$.
21. $(m+5)(m-3)$.
22. $(x-11)(x+7)$.
23. $(x+3)(x+20)$.
24. $(x-2)(x+15)$.
25. $(m-4)(m-6)$.
26. $(a-3)(a-8)$.
27. $(p+4)(p-11)$.
28. $(t-6)^2$.
29. $(a+5)(a+9)$.
30. $(a-7)(a+3)$.
31. $(x+2)(x+4)$.
32. $(x+7)(x-9)$.
33. $(x-4)(x-9)$.
34. $(x+2)(x-10)$.
35. $(x-5)^2$.
36. $(m+3)(m-12)$.
37. $(x-y)(x-2y)$.
38. $(p-5q)(p-7q)$.
39. $(a-5b)(a+8b)$.
40. $(b-4c)^2$.
41. $(p-5q)(p+10q)$.
42. $(x+6y)(x-8y)$.
43. $(a+5)(a-5)$.
44. $(x+4)(x-4)$.
45. $(x+3)(x-3)$.
46. $(ab-9)(ab+10)$.
47. $(xy-9)(xy+12)$.
48. $(pq-8)(pq+7)$.
49. $(x+1)(2x+1)$.
50. $(x+2)(2x+1)$.
51. $(x+1)(2x+3)$.
52. $(4y+1)(y+1)$.
53. $(a-2)(3a-1)$.
54. $(a+2)(5a-1)$.
55. $(x+7)(2x+1)$.
56. $(x-1)(5x-3)$.
57. $(x+1)(3x-5)$.
58. $(3x+5)(2x-1)$.
59. $(3x+2)(x-1)$.
60. $(y+1)(3y+5)$.
61. $(x+2)(7x+2)$.
62. $(5x-y)(x+y)$.
63. $(3x+4)(2x-1)$.
64. $(4a+1)(2a-1)$.
65. $(2a+5)(2a-5)$.
66. $(x+2)(2x+3)$.
67. $(3x+2)(2x+3)$.
68. $(3x+2)(2x-3)$.
69. $(x+1)(8x-5)$.
70. $(5x+4)(2x-3)$.
71. $(3x+5)(x+2)$.
72. $x(4x-5)$.
73. $(4x-3)(2x+3)$.
74. $(4x+1)(2x+9)$.
75. $(6x-5)(2x-1)$.
76. $(7a-3)(2a+5)$.
77. $(3x+2)(2x-5)$.
78. $(4a-7)(4a+7)$.
79. $(x-3)(3x-11)$.
80. $(4a-5b)(a-2b)$.
81. $(3x-4y)(4x-3y)$.
82. $(4x+3y)(3x-4y)$.
83. $(4p+q)(3p+2q)$.
84. $(5x-3)(3x-5)$.
85. $(4x-7y)(2x+y)$.
86. $(2ab+3)(2ab-7)$.
87. $(4ab+7)(3ab+4)$.
88. $(2x^2-3)^2$.
89. $(2a^2+7)(a^2-3)$.
90. $(4+5x)(3-2x)$.
91. $(7-x)(6-5x)$.
92. $(3-x)(11+2x)$.
93. $(3-8t)(1-4t)$.
94. $(2-p^3)^2$.
95. $(2x-y^2)(x-10y^2)$.
96. $(3x^2-y)(2x^2+3y)$.
97. $a+3$. 98. $2x+1$.
99. $y+3$. 100. $\frac{a-7}{a-1}$.
101. $\frac{x-1}{x+2}$.
102. $\frac{y+9}{y-3}$.
103. $\frac{1}{x+1}$.
104. $\frac{x}{x-10}$.
105. $\frac{2x+1}{x-1}$.
106. $\frac{2a-3}{a+2}$.
107. $\frac{a-3}{a-1}$.
108. $\frac{2a+5}{a+3}$.
109. $\frac{3t+7}{t}$.
110. $\frac{3x+y}{2x-3y}$.
111. $\frac{a+2b}{a+5b}$.
112. 8. 113. 11.
114. 20. 115. 35.
118. $(2x+1)(x-2)$.
119. $(x-3)(x+7)$.
120. $(3x-1)(x+4)$.
121. $x+2$.

EXAMPLES XIII (p. 84).

1. 5, 7.
2. 4, 10.
3. 3, −6.
4. −1, −4.
5. −8, 7.
6. −11, −10.
7. 0, 3.
8. 0, −8.
9. a, $-b$.
10. $\frac{1}{2}$, $3\frac{1}{2}$.
11. $-\frac{2}{3}$, $4\frac{1}{2}$.
12. $2\frac{1}{4}$, $-1\frac{1}{3}$.
13. $-2\frac{1}{5}$, $1\frac{1}{2}$.
14. 0, $4\frac{1}{3}$.
15. $\frac{m}{2}$, $\frac{n}{3}$.
16. $-\frac{3m}{4}$, $\frac{4n}{3}$.
17. $\frac{-6p}{5}$, $\frac{-p}{2}$.
18. $-\frac{a}{2}$, $-\frac{a}{2}$.
19. 1, 3.
20. 2, 3.
21. −2, 1.
22. 5, 8.
23. 9, −5.
24. 2, −7.
25. 0, 9.
26. 3, −3.
27. 3, −8.
28. 1, −4.
29. 2, 5.
30. 0, 7.
31. 3, 6.
32. 4, 4.
33. 4, 16.
34. 3, −15.
35. 8, −9.
36. 12, −2.
37. 4, 6.
38. 4, −12.
39. 18, −3.
40. 8, −16.
41. $\frac{1}{2}$, 1.
42. $\frac{2}{3}$, 1.
43. $1\frac{1}{2}$, −2.
44. 1, $-3\frac{1}{2}$.
45. 2, $-\frac{1}{3}$.
46. $\frac{1}{2}$, 4.
47. 7, $-\frac{1}{3}$.
48. $\frac{1}{5}$, −1.
49. $\frac{1}{5}$, 2.
50. $2\frac{1}{2}$, 1.
51. 3, $-\frac{1}{3}$.
52. $1\frac{2}{3}$, $-\frac{1}{2}$.
53. $2\frac{1}{2}$, $2\frac{1}{2}$.
54. $1\frac{1}{3}$, $-1\frac{1}{2}$.
55. $1\frac{1}{2}$, $\frac{1}{4}$.
56. 0, $4\frac{2}{4}$.
57. 4, $-2\frac{1}{3}$
58. $-2\frac{1}{2}$, $-4\frac{1}{2}$.
59. $\frac{1}{2}$, $-2\frac{3}{4}$.
60. $2\frac{1}{2}$, $-1\frac{2}{3}$.
61. $\frac{2}{3}$, $-1\frac{2}{3}$.
62. $2\frac{2}{3}$, −2.
63. $\frac{1}{3}$, $-\frac{1}{3}$.
64. $-1\frac{1}{2}$, $-2\frac{1}{2}$.
65. 0, 5.
66. $-1\frac{1}{2}$, $3\frac{1}{2}$.
67. $\frac{1}{6}$, 1.
68. $\frac{1}{3}$, $\frac{1}{3}$.
69. $\frac{2}{3}$, $-4\frac{1}{2}$.
70. 2, $-3\frac{1}{2}$.
71. $\frac{2}{5}$, $-2\frac{1}{2}$.
72. 8, $-1\frac{1}{2}$.
73. 9, −4.
74. 0, $3a$.
75. $2a$, $-9a$.
76. $5a$, $-\frac{a}{2}$.
77. 16, −15.
78. 1, 2.
79. 0, $2\frac{1}{3}$.
80. 1, −2.
81. 2, $-\frac{2}{5}$.
82. 6, $1\frac{1}{2}$.
83. 16, 0.
84. $-\frac{1}{3}$ or $3\frac{2}{3}$.
88. 14; 12.
89. 7, 12.
90. 3, 7.
91. 2, 3, 4.
92. 13 in., 4 in.
93. 7 in., 8 in.
94. 4 in., 8 in.
95. 3, 13.
96. $(8 - 2x)$ in., $(6 - 2x)$ in.; $\frac{1}{2}$ in.
97. 12 in., 8 in.
98. 3 in.
99. 5 cm., 12 cm.
100. 8 ch. by 6 ch.
101. 1 yd.
102. 6, −2.
103. 2, $\frac{5}{6}$.
104. $\frac{1}{2}$ sec., 2 sec.

EXAMPLES XIV (a) (p. 88).

1. $2(x + 2y)$.
2. $3(a - 2b)$.
3. $5(2x + 1)$.
4. $p(q - 1)$.
5. $x(a - b)$.
6. $7(a + 2b - 3c)$.
7. $3a(x + 4y)$.
8. $x^3(x + 2)$.
9. $2x(x^2 + 1)$.
10. $2a(a + 2b - 5c)$.
11. $r^2(1 + r + r^2)$.
12. $4p^2q(1 + 2q)$.
13. $3x^2(1 - 5y)$.
14. $3(x + 1)(x + 3)$.
15. $2(x - 3)(x + 2)$.
16. $x(x + 3)(x - 2)$.
17. $5(x - 3)(x - 4)$.
18. $\frac{h}{2}(a + b)$.
19. $\frac{22r}{7}(r + 2h)$.
20. $4n(3m - 5n)(m - n)$.
21. $x(a - b + c)$.
22. $s(x + a - b)$.
23. $p^2(x - y - z)$.
24. $3x(a - b + c)$.
25. $(p - q)(x - y)$.
26. $(a + b)(4m - 3n)$.
27. $5(x + y)(b - c)$.
28. $(3x + 4)(x + y - 5)$.
29. $(a + b)(x - y + z)$.
30. $(a + b)(a + 2b)$.
31. $2x(x + y)$.
32. $3a(a + 2b)$

33. 4300.
34. 1700.
35. $a + 3x$.
36. $a + 2$.
37. $p + 4$.
38. $x + 3$.
39. $\frac{2}{3}$.
40. $\frac{5}{x}$.
41. $\frac{5a}{4}$.
42. $\frac{x-8}{x}$.
43. $\frac{m-2}{2n}$.
44. $\frac{p+s}{s+t}$.

EXAMPLES XIV (b) (p. 89).

1. $(a + 3)(b + 2)$.
2. $(a + b)(c + d)$.
3. $(a - 2)(b - 1)$.
4. $(a - 4)(3x + 1)$.
5. $(x - y)(3x + 7)$.
6. $(x - a)(x - b)$.
7. $(p - q)(x^2 + y^2)$.
8. $(p - q)(q - r)$.
9. $(a - 1)(a^2 + 1)$.
10. $(m - n)(m + 1)$.
11. $(m - n)(m - 1)$.
12. $(a + c)(b - d)$.
13. $(m - 2n)(a - b)$.
14. $(2q - s)(p - r)$.
15. $(a - x)(b - x)$.
16. $(2 - 3l)(3l^2 + m)$.
17. $(a - 2)(a^2 + 1)$.
18. $(a^2 + 1)(a^3 - 4)$.
19. $(x + 2y^2)(2x^2 + y)$.
20. $(m^2 - 2)(l^2 + 1)$.
21. $(bx - a)(ax + b)$.
22. $(y - c)(y - 5c)$.
23. $(a - b)(x + y + z)$.
24. $(x + y)(a + b - c)$.
25. $(2p + q)(a - b + 1)$.
26. $(m + n)(3 + m + n)$.
27. $(m - n)(4 - m + n)$.
28. $a + b - 2x - a - b$.
29. $(p + 2q)(p + 2q - 5)$.
30. $(y + a)(y - 3)$.
31. $(x + y)(x - y - 4)$.
32. $(x - 2y)(x + 2y - 4)$.
33. $(x - 1)(x - 2 + a)$.
34. $(x + z)(y - a)$.
35. $x - 2y$.
36. $x - 2y$.
37. $x^2 - 2$.

EXAMPLES XIV (c) (p. 91).

1. $(x + 2)(x - 2)$.
2. $(x + 3)(x - 3)$.
3. $(y + 5)(y - 5)$.
4. $(y + 7)(y - 7)$.
5. $(6 + a)(6 - a)$.
6. $(1 + m)(1 - m)$.
7. $(p + 9)(p - 9)$.
8. $(2a + 3)(2a - 3)$.
9. $(3x + 1)(3x - 1)$.
10. $(10 + m)(10 - m)$.
11. $(4p + 5)(4p - 5)$.
12. $(xy + 6)(xy - 6)$.
13. $(11b + 3c)(11b - 3c)$.
14. $(4 + ab)(4 - ab)$.
15. $(10x + 7y)(10x - 7y)$.
16. $(a^2 + 3)(a^2 - 3)$.
17. $(6 + a^2b)(6 - a^2b)$.
18. $(x^2 + y^2)(x + y)(x - y)$.
19. $(4m^2 + 9)(2m + 3)(2m - 3)$.
20. $(a^2b^2 + 4)(ab + 2)(ab - 2)$.
21. $(1 + 4x^2y^2)(1 + 2xy)(1 - 2xy)$.
22. $3(a + 3)(a - 3)$.
23. $5(1 + x)(1 - x)$.
24. $x(a + b)(a - b)$.
25. $b(a + 2b)(a - 2b)$.
26. $2(x + 5y)(x - 5y)$.
27. $3(9 + x^2)(3 + x)(3 - x)$.
28. $(a + b + c)(a + b - c)$.
29. $(a - b + c)(a - b - c)$.
30. $(3 + x + y)(3 - x - y)$.
31. $(x + y - z)(x - y + z)$.
32. $(x + 5 + 3)(x + 5 - 3)$.
33. $(9 - x)(1 + x)$.
34. $(3x + 10)(3x - 8)$.
35. $x(14 - x)$.
36. $(2a + 2b + c)(2a + 2b - c)$.
37. $(3x - 6y + 2z)(3x - 6y - 2z)$.
38. $(10 + 11p - 11q)(10 - 11p + 11q)$.
39. $(5a + 4b)(a - 2b)$.
40. $3p(p - 2q)$.
41. $(7a + 2b)(a + 12b)$.
42. $8(7x - 6y)(2x - y)$.

43. $(8a + 7b)(2a - 7b)$.
44. $4(1 - x)(1 + 2x)$.
45. $(x + 2)^2(x + 3)(x - 3)$.
46. 200.
47. 14.
48. 160,000.
49. 307·4.
50. 172 sq. in.
51. 12 inches.
52. 1150.
54. $\frac{y}{x-2}$.
55. $\frac{x+3}{2x+4}$.
56. $\frac{a+12}{5}$.
57. $\frac{2m+3}{2m-3}$.
58. 12, 1.

59. (i) 16 in., 5 in., 126 sq. in.; (ii) 8 in., 20 in., 210 sq. in.; (iii) 16 in., 5 in., 66 sq. in.

63. $(a + b + c)(a + b - c)$.
64. $(a + b - 1)(a - b - 1)$.
65. $(2x + y + z)(2x + y - z)$.
66. $(x + y - z)(x - y + z)$.
67. $(3a + b + c)(3a - b - c)$.
68. $(4a + 3p + q)(4a - 3p - q)$.
69. $(x + 5y - z)(x - 5y + z)$.
70. $(5a + c + 1)(5a - c - 1)$.
71. $(6 + 2a + 3b)(6 - 2a - 3b)$.
72. $(l + 2m - n)(l - 2m + n)$.
73. $(a + b - 1)(a - b + 1)$.
74. $(3a + c - s)(3a - c - s)$.
75. $(p + q + r - s)(p + q - r + s)$.
76. $(k - l + m - n)(k - l - m + n)$.
77. $(x - y)(x + y - 9)$.
78. $(x + y)(x - y - 9)$.
79. $(p + q)(p - q - 3)$.
80. $(x - y)(x + y - 4)$.
81. $(a - b)(a + b - 1)$.
82. $(x + y)(a - x + y)$.
83. $(m + n)(5 - m + n)$.
84. $(a+b)(a-b)(a^2+b^2+x^2)$.
85. $-2m(4 + m)$.

EXAMPLES XIV (d) (p. 93).

1. $(x - 1)(x^2 + x + 1)$.
2. $(a + 1)(a^2 - a + 1)$.
3. $(1 - a)(1 + a + a^2)$.
4. $(1 + x)(1 - x + x^2)$.
5. $(x + 2)(x^2 - 2x + 4)$.
6. $(2 - x)(4 + 2x + x^2)$.
7. $(y - 3)(y^2 + 3y + 9)$.
8. $(4 - x)(16 + 4x + x^2)$.
9. $(2m + 1)(4m^2 - 2m + 1)$.
10. $(x - 5y)(x^2 + 5xy + 25y^2)$.
11. $(3x - a)(9x^2 + 3ax + a^2)$.
12. $(2xy + 3)(4x^2y^2 - 6xy + 9)$.
13. $(10 - m)(100 + 10m + m^2)$.
14. $(2l - 5m)(4l^2 + 10lm + 25m^2)$.
15. $(6 - pq)(36 + 6pq + p^2q^2)$.
16. $(7x - 3y)(49x^2 + 21xy + 9y^2)$.
17. $(4m + 5n)(16m^2 - 20mn + 25n^2)$.
18. $(9 - 2a)(81 + 18a + 4a^2)$.
19. $3(n + 3)(n^2 - 3n + 9)$.
20. $4(x + 1)(x^2 - x + 1)$.
21. $m(m - 1)(m^2 + m + 1)$.
22. $5(c - 2d)(c^2 + 2cd + 4d^2)$.
23. $2(1 - 5a)(1 + 5a + 25a^2)$.
24. $x^2(a - b)(a^2 + ab + b^2)$.
25. $(a^2 + b^2)(a^4 - a^2b^2 + b^4)$.
26. $(a^2 - 2b^2)(a^4 + 2a^2b^2 + 4b^4)$.
27. $(4p^2 - 3)(16p^4 + 12p^2 + 9)$.
28. $(a + b)(x + y)(x^2 - xy + y^2)$.
29. $(a + b)^3$.
30. $(a - b)^3$.
31. $x^2 + 1$.
32. $2x - 1$.
33. $\frac{x^2 - x + 1}{(x + 1)^2}$.
34. $\frac{x^2 + 2x + 4}{x + 2}$.
35. $\frac{(x - 3)^2}{x^2 + 3x + 9}$.
36. $\frac{x - 2}{4x^2 - 2x + 1}$.
37. $\frac{x^2 - 3x + 9}{x + 9}$.
38. $\frac{a^2 + 2a + 4}{a + 5}$.

REVISION EXAMPLES (p. 94).

1. $2x(x+4)$.
2. $2(x+1)(x+3)$.
3. $ab(b+a)(b-a)$.
4. $3(x+10)(x-10)$.
5. $(x-3)(x-13)$.
6. $(3-x)(1+4x)$.
7. $(a+2x)(m-n)$.
8. $(a+2)(a-2)(a+3)(a-3)$.
9. $(3+x)(a+b)(a-b)$.
10. $x(x^2+1)(x+1)(x-1)$.
11. $(ab-5)(ab-6)$.
12. $(7+x)(1-x)$.
13. $(3p-10)(p+1)$.
14. $(x-1)(x-4)$.
15. $x(x+2)(x+9)$.
16. $(5x-3y)(2x+y)$.
17. $\left(\frac{a}{2}+10\right)\left(\frac{a}{2}-10\right)$.
18. $(m+3n+1)(m+3n-1)$.
19. $2(x+16)(x-3)$.
20. $(a+b)(a+b+5)$.
21. $(5a+9)(3a-4)$.
22. $(8-3x)(4+x)$.
23. $5(p-q)(5p-6q)$.
24. $m(2l-m)$.
25. $8ax$.
26. $(x+1)(x+y)$.
27. $(3x+8y)(2x-7y)$.
28. $(n-1)(m-1)$.
29. $(3x^2+7x+6)(x^2-x-4)$.
30. $(3x+y)(3x-y)(x+3y)(x-3y)$.
31. $2(x+4)(3x+4)$.
32. $(4x+3y)(m-2)$.
33. $x(2x+b)(2x-b)$.
34. $(q-1)(p-q-1)$.
35. $3(5m+1)(m-5)$.
36. $(p+1)(p-3)(p^2-2p+3)$.
37. $(x+1)(x-1)(x+2)(x-2)$.
38. $(x-1)^2(x+1)$.
39. $(a+b)(x+2)^2$.
40. $(3x+y+z)(3x-y-z)$.
41. $(5-2x)(3-x)$.
42. $(x+3y)(x-3y)(x^2+9y^2)$.
43. $7(x+3c)(x^2-3cx+9c^2)$.
44. $(a+b)(a+b+2)(a+b-2)$.
45. $(a-x)(a+x-9)$.
46. $(a^2+b^2)(x^2+y^2)$.
47. $(x-a)(x+2b)$.
48. $(3x-y)(3x+y-5)$.
49. $p(q-4)(q^2+4q+16)$.
50. $(3a+4b+c)(3a+4b-c)$.
51. $(a+b+1)^2$.
52. $4(b-7)(b-13)$.
53. $(x-1)(x-2)(x+2)(x+3)$.
54. $(a+b)(a^2+b^2)$.
55. $(a-b)(a+b+ab)$.
56. $(a+q)(p-r)$.
57. $(p+q)^2(p-q)(p^2-pq+q^2)$.
58. $4(a-4)$.
59. $(a-b)^3$.
60. $(a-b)(x^2+ab)$.
61. $(x+2a)(x-2a-b)$.
62. $(3x-2y)(x+8y)$.
63. $(8a-5b-3c)(2a-5b+3c)$.
64. $(a+1)(a-2)(a+3)(a-4)$.
65. $(2x+5)(x+3)$.
66. $11(x-y)(13x-5y)$.
67. $x(x+2y)(x+6y)(x+8y)$.
68. $(m+2n)(m+2n+3)(m+2n-3)$.
69. $(x+2)(x-5)$.
70. $(a+b+c)^2$.
71. $2(a^2+b^2)$. 72. 0.
74. $12x(x+2)(x-2)$; $2x(x-2)$.
75. 4, $\frac{1}{2}$.
76. $(x+1)(x+2)(x+3)$. 77. 1.
78. $(a+1)^2(a+4)$.
79. $(4x+3)(x-1)$. 80. -4.
81. a^2+1.
82. $x+1$, and x^2+x+1.
83. $(x-3)(x+9)(x-7)$.

EXAMPLES XV (p. 96).

1. $\frac{x}{y}$.
2. $\frac{b}{a^2}$.
3. ab.
4. $\frac{p^2}{32}$.
5. $\frac{5x}{8}$.
6. $\frac{2b}{3ax}$.
7. $\frac{4ab}{5}$.
8. $\frac{2ab^2}{3c}$.
9. $\frac{5a^2x}{4b}$.
10. $\frac{2mq}{9n}$.
11. $\frac{16}{9}$.
12. $\frac{x-7}{21}$.
13. $2y(x-2)$.
14. $1\frac{1}{5}$.
15. $\frac{2x(x-1)}{x-2}$.
16. $\frac{x(x+2)}{x+1}$.
17. $\frac{5b^2}{a-b}$.
18. 1.
19. $\frac{q}{10r}$.
20. $\frac{2x}{15}$.
21. $m-3$.
22. $\frac{2(x-4)}{(x+2)(x+5)}$
23. $x+1$.
24. $\frac{x+3}{x-3}$.
25. $\frac{p-6}{p-3}$.
26. $\frac{a^2+2a}{2a^2+2}$.
27. $\frac{x-1}{2x-1}$.
28. $\frac{a(b+c)}{bc(a^2+b^2)}$.
29. $\frac{a}{b}$.
30. $\frac{15}{2x+3y}$.
31. $\frac{-a(a+1)}{a+2}$.
32. $\frac{x(x+y)}{u}$.

EXAMPLES XVI (p. 99.)

1. a^2.
2. $6a^2$.
3. abc.
4. x^3y^3.
5. $12x^3$.
6. $40ab^2$.
7. $30xyz$.
8. $20y^5$.
9. $12a^3b$.
10. $60\ a^3b^2c^2$.
11. $6x(x-1)$.
12. $24(x-2)(x-3)$.
13. $x(x-a)(x-b)$.
14. $x^2(x-2)(x-5)^2$.
15. $(x-1)(x-2)(x-3)$.
16. $(m-4)(m+6)^2$.
17. $12(x-y)(x+y)$.
18. $(a-b)(a+b)^2$.
19. $18a(a-b)^2$.
20. $5x(x+5)(x+8)$.
21. $(x-2)(x-3)(3x-1)$.
22. $x(3x+4y)(3x-4y)$.
23. $10pq(p+2q)(p-q)$
24. $m(m+4n)(m-4n)^2$.
25. $(2x-1)^2(x-2)^3$.
26. $ab^2(a-5)^2(a-12)$.
27. $12m^2n(m-n)(m-4n)$.
28. $60(a+x)^2(a-x)(a^2+x^2)$.
29. $24x(x+a)^3(x-a)$.
30. $24(x+y)(x-6y)(x-4y)$.

EXAMPLES XVII (p. 102).

1. $\frac{13x}{12}$.
2. x.
3. $\frac{x}{28}$.
4. $\frac{a+b}{3}$.
5. $\frac{78p-13q}{36}$.
6. $\frac{7x+5}{6}$.
7. $-\frac{x}{2}$.
8. $\frac{5m-9}{24}$.
9. $\frac{7}{8}$.
10. $\frac{x+3}{28}$.
11. $\frac{5x+y}{6}$.
12. $\frac{5a-5x}{12}$.
13. $\frac{-3y}{10}$.
14. $\frac{5a+13b}{12}$.
15. $\frac{5}{6x}$.
16. $\frac{a}{2x}$.

17. $\frac{5}{4y}$.
18. $\frac{2a^2-3a+1}{a^2}$.
19. $\frac{3y^2-4xy+2x^2}{x^2y^2}$.
20. $\frac{5ab-4a^2}{2b^2}$.
21. $\frac{3c+4a}{abc}$.
22. $\frac{p+4}{30x}$.
23. $\frac{1+6p}{6p}$.
24. $\frac{8x-3}{4x^2}$.
25. $-\frac{1}{15x}$.
26. 0.
27. $\frac{pq-3p^2-4q^2}{6p^2q^2}$.
28. $\frac{3x+1}{x(x+1)}$.
29. $\frac{x+4}{(x+1)(x+2)}$.
30. $-\frac{14}{(2x+1)(x-3)}$.
31. $\frac{a^2+b^2}{(a+b)(a-b)}$.
32. 0.
33. $\frac{3b}{(a-b)^2}$.
34. $\frac{x(a-b)}{(x+a)(x+b)}$.
35. $-\frac{2a^2}{(a-3b)(a-5b)}$.
36. $\frac{a}{(a-1)^2}$.
37. $\frac{2}{(x+1)(x-1)}$.
38. $\frac{5}{xy}$.
39. $\frac{6}{(x+4)(x-2)}$.
40. $\frac{1}{a-2b}$.
41. $\frac{2}{(x-1)(x-2)(x-3)}$.
42. $\frac{1}{(x-2)(x-3)}$.
43. $\frac{2}{(a-1)(a-2)}$.
44. $\frac{2}{2x+3y}$.
45. 0.
46. $\frac{5}{6(x-3)}$.
47. 0.
48. $\frac{1}{4a-1}$.
49. 0.
50. $\frac{3}{a-2}$.
51. 0.
52. $\frac{2}{2x+1}$.
53. $\frac{9+6x}{(2+x)(1+x)}$.
54. $\frac{6-4x}{(x-1)(x-2)(x-3)}$.
55. $\frac{y}{(x-y)(x-2y)}$.
56. $\frac{4ab}{a^2-b^2}$.
57. $\frac{1}{x-1}$.
58. $\frac{2}{x-4}$.
59. $\frac{x+1}{x+3}$.
60. $a+b$.
61. $\frac{10}{2x-3}$.
62. $\frac{2xy}{(x+y)^2(x-y)}$.
63. $\frac{2y^2}{(x+y)^3}$.
64. $\frac{1}{xy}$.
65. $\frac{x+1}{x-1}$.
66. $\frac{2b-a}{2ab}$.
67. $\frac{11}{8(a-b)}$.
68. $-\frac{2+3a}{(a+2)(a-2)}$.
69. $\frac{2}{x+1}$.
70. $-\frac{(x+5)}{(x-5)(x-10)}$.
71. $\frac{7}{(p-1)(p-2)}$.
72. $\frac{4}{(x+1)(x-1)(x+2)(x-2)}$.
73. $\frac{1}{(x-2)(x-3)(x-4)}$.
74. $-\frac{1}{(m-5)(m-6)}$.
75. 4.
76. 1.
77. $-\frac{16a^3x}{16a^4-x^4}$.
78. $\frac{4x}{7(x+5)}$.
79. $\frac{y}{x}$.

EXAMPLES XVIII (p. 109).

1. ± 12.
2. ± 20.
3. $\pm$.
4. $\pm 2\frac{1}{2}$.
5. 7 or -3.
6. 6 or -14.
7. 6 or 4.
8. $-\frac{1}{2}$ or $-6\frac{1}{2}$.
9. $8\frac{1}{2}$ or $7\frac{1}{2}$.
10. $-\frac{5}{9}$ or $-\frac{1}{9}^3$.
11. 6·41 or 3·59.
12. $-$ ·27, $-$ 3·73.
13. $a+b$ or $a-b$.

14. $a - 2b$ or $-a - 2b$.
15. $p + \frac{q}{2}$ or $p - \frac{q}{2}$.
16. $x^2 + 4x + 4$.
17. $x^2 - 10x + 25$.
18. $x^2 - 16x + 64$.
19. $x^2 + 20x + 100$.
20. $x^2 + 5x + \frac{25}{4}$.
21. $x^2 - 7x + \frac{49}{4}$.
22. $x^2 - x + \frac{1}{4}$.
23. $x^2 + \frac{2}{3}x + \frac{1}{9}$.
24. $x^2 - \frac{4}{5}x + \frac{4}{25}$.
25. $x^2 + \frac{1}{3}x + \frac{1}{36}$.
26. $x^2 - \frac{3}{4}x + \frac{9}{64}$.
27. $x^2 + 3ax + \frac{9a^2}{4}$.
28. 1 or -3.
29. 13 or -3.
30. 11 or 1.
31. 3 or 15.
32. 8 or -10.
33. 14 or -12.
34. 40 or -6.
35. 5 or -8.
36. 12 or -7.
37. 5 or -12.
38. 9 or -5.
39. 1·45 or $-3·45$.
40. 5·32 or $-1·32$.
41. 10·29 or $-·29$.
42. 5·24 or ·76.
43. $-5·83$ or $-·17$.
44. $-5·41$ or $-2·59$.
45. ·45 or $-4·45$.
46. 11·63 or $-1·63$.
47. 13·21 or $-1·21$.
48. 14·14 or $-·14$.
49. 5·54 or $-·54$.
50. 8·41 or ·59.
51. ·21 or $-1·21$
52. 7·11 or $-·11$.
53. 1·24 or $-3·24$.
54. 1·27 or $-6·27$.
55. $\pm 2·45$.
56. 2·87 or $-·87$.
57. 3·44 or $-·44$.
58. 2 or $-\frac{1}{2}$.
59. ·66 or $-2·66$.
60. 6·06 or $-·06$.
61. ·58 or $-4·58$.
62. ·93 or $-5·93$.
63. ·37 or $-1·37$.
64. ·82 or $-2·42$.
65. ·80 or $-1·55$.
66. $\frac{2}{3}$ or $-\frac{1}{2}$
67. $1\frac{1}{3}$ or $-2\frac{1}{2}$.
68. ·85 or $-2·35$.
69. 1·05 or $-2·38$.
70. 1·08 or $-3·41$.
71. 3 or $-\frac{2}{3}$.
72. $2\frac{2}{3}$ or -1.
73. 1·16 or 2·59.
74. 1·93 or $-·26$.
75. 4·05 or $-5·55$.
76. ·51 or 2·82.
77. 6·54 or ·46.
78. 1·31 or $-10·31$.
79. 1·56 or $-2·56$.
80. 5 or -6.
81. 2·44 or $-1·44$.
82. $-·20$ or $-1·13$.
83. 4 or $\frac{1}{2}$.
85. 4·71 or 3·29.
88. $4\frac{2}{3}$ or 3.
89. 3·58 in., 5·58 in.
90. 5·41 in., 2·59 in.
91. 3·73 or ·27.
92. 5·4 in.
93. The segments are 14 in., 9 in.
95. 9·74 in., 2·26 in.

96. AP = 2 in., BP = 16 in., or AP = 7 in., BP = 11 in.

97. 7·37 in. and 1·63 in.
98. 30 m.p.h., 25 m.p.h.
99. 10d.
100. 13 ft., 26 ft.
101. 1·5 sec., 8·5 sec.
103. 6 in. from either end.
104. 2 ft., 32 ft.
105. 8 or -3.
106. 6·57 or $-106·57$.
107. 20 oxen, £12 each.
108. 3/6.
109. A, 4 m.p.h.; B, $3\frac{3}{4}$ m.p.h.
110. 15 in., 36 in., 39 in.
111. 40 m.p.h., 30 m.p.h.
112. 6/-.
113. 8·54 or 1·46.
115. 80 ft., 120 ft.

116. Sides of rect., 4 in. and 12 in.; side of square, 13 in., or Sides of rect., 8 in. and 24 in.; side of square, 5 in.

117. 4 m.p.h.
118. 5·46 in., 3·24 in.
119. 7 ft., 18 st.
120. 11 in.
121. 100 shares, £75 per share.
122. 5·42 in.

EXAMPLES XIX (a) (p. 115).

1. $-\frac{5}{3}b$. 2. $\frac{b}{a}$. 3. 1.
4. a. 5. $a-b$. 6. $\frac{5pq}{p-q}$.
7. q. 8. n. 9. $c+d$.
10. $\frac{5a+2b}{3}$. 11. 0. 12. $-\frac{2cd}{c-d}$.
13. $2(a+b)$. 14. $a-b$. 15. $a+b$.
16. $\frac{a}{2}$. 17. $-\frac{1}{2\rho}$. 18. $\frac{ab}{a-b}$.
19. $\frac{ac}{b+c}$. 20. 1. 21. a.
22. $b-a$. 23. $b-a$. 24. $\frac{a+b}{2}$.
25. $\frac{p}{6}$ in., $\frac{p}{3}$ in.
26. $\frac{a+b}{2}$ in., $\frac{a-b}{2}$ in.
27. $\frac{2s-a}{4}$ cm., $\frac{6s-3a}{4}$ cm.
28. $\frac{180m^\circ}{m+2}$, $\frac{180^\circ}{m+2}$, $\frac{180^\circ}{m+2}$.
29. £$\frac{p}{n+1}$, £$\frac{np}{n+1}$.
30. £$\frac{m+n}{2}$, £$\frac{m+n}{4}$, £$\frac{n-3m}{4}$.
31. $\frac{3mn}{m+n}$ hr.
32. $\frac{n}{m-n}$.
33. $\frac{5b+c}{a-b}$.
34. $\frac{a}{3}$, $2a$.
35. $-\frac{b}{4}$, $\frac{3b}{2}$.
36. $\frac{3a}{2}$, $-2a$.
37. $a+b$, $a-b$.
38. $2a$, b.
39. $-m$, n.
40. $\frac{p}{3}$, $\frac{3q}{p}$.
41. $1, \frac{b}{a}$.
42. $\frac{a}{b}$, $\frac{b}{a}$.
43. $a, \frac{a+b}{2}$.
44. b, $-b$.
45. $3a$, $11a$.

EXAMPLES XIX (b) (p. 116).

1. $4a$, a.
2. $3c$, c.
3. c, $-c$.
4. $\frac{3b}{a}$, b.
5. pq, q.
6. $-9n$, $10m$.
7. a, b
8. $a+b$, $a-b$.
9. l, $-l$.
10. $p+q$, 0.
11. $2p$, $2q$.
12. a, b.
13. a^2, 1.
14. $\frac{a^2}{a+b}$, $-\frac{a^3}{a+b}$.
15. $\frac{a^2+b^2}{2ab}$, $-\frac{a^2-b^2}{2b}$.
16. $\frac{a^3+b^3}{a^2+b^2}$, $\frac{ab(a-b)}{a^2+b^2}$.
17. $-\frac{b}{1+m}$, $\frac{b}{1+m}$.
18. $\frac{m+n}{2}$, $\frac{m-n}{2}$.
19. $c+d$, $c+d$.
20. $-\frac{c(b_1-b_2)}{a_1b_2-a_2b_1}$, $\frac{c(a_1-a_2)}{a_1b_2-a_2b_1}$.

EXAMPLES XIX (c) (p. 117).

1. $\frac{2A}{b}$.
2. $\frac{P-2b}{2}$.
3. $\frac{40}{p}$.
4. $\frac{6-l}{2\pi}$.
5. $2s-b-c$.
6. $\frac{P}{\pi+2}$.
7. $\frac{c}{y-a}$.
8. $my-m^2x$.
9. $\frac{wv^2}{gF}$.
10. $\frac{2s-vt}{t}$.
11. $\frac{s-a}{s}$.
12. $\frac{3y-2}{1-y}$.

13. $\frac{g+a}{g-a}$. Q. 14. $\frac{V-v}{vt}$. 21. $\frac{b-c}{b-c-y}$. 22. $\frac{v^2-u^2}{2a}$.
15. $\frac{100}{P}(A-P)$. 16. $\frac{y^2}{4a}$. 23. $\frac{g}{W}(F-2)$. 24. $\frac{vf}{v-f}$.
17. $\frac{b-a}{y}$ 18. $\frac{y}{c-a}$. 25. $\frac{16wt^2}{16t^2-h}$. 26. $\frac{vr}{nr-nv+v}$.
19. $\frac{b^2+c^2-a^2}{2bc}$. 20. $-\frac{fy+c}{y+g}$.

EXAMPLES XX (p. 125).

6. 5·76; ± 1·41. 7. 2·2; 3·17. 8. −2·25; 2·62, ·38.
9. $y = 4$, $x = 2$; 3·41, ·59; 4·45, −·45; 4·83, −·83.
10. $y = -2·25$, $x = ·5$; 1·62, −·62. 11. $y = 6·93$; 5·19, 2·31.
12. $\frac{1}{2}$. 13. 39·2; −12·2, 13·3. 14. $9\frac{1}{2}$, $3\frac{1}{3}$, $1\frac{1}{3}$.
15. 1·75, 1·13. 16. −2·4, −2·2. 17. 4·19, −1·19.
18. −19, −2; ·55, −4·55; −3·41, −·59. 19. 2·3 or −1·3.
20. 5·24 or ·76. 21. 3·37 or −2·37, $x^2 - x - 8 = 0$.
22. −3·30, ·30, 3. 23. 3·65, −1·65.
24. −1·46, 2, 5·46. 25. −·9, 1·2, 3·7.

EXAMPLES XXI (a) (p. 131).

1. $(4, \frac{1}{2})$, $(-1, -2)$.
2. $(2, 2)$, $(-1, -1)$.
3. $(10, 4)$, $(8, 5)$.
4. $(1, 2)$, $(2, 1)$.
5. $(5, 0)$, $(-4, 3)$.
6. $(2, 1)$, $(-4\frac{2}{3}, 4\frac{1}{3})$.
7. $(4, 2)$, $(\frac{1}{4}, \frac{1}{2})$.
8. $(4, 3)$, $(-1\frac{2}{3}, 8\frac{2}{3})$.
9. $(1, 3)$, $(-1\frac{4}{5}, 2\frac{3}{5})$.
10. $(3, 2)$, $(27, -22)$.
11. $(3, 1)$, $(5, -3)$.
12. $(3, -2)$, $(2, -3)$.
13. $(3, 2)$, $(-1, 6)$.
14. $(2, 1)$, $(\frac{5}{6}, -2\frac{1}{2})$.
15. $(2, 4)$, $(30, 116)$.
16. $(-1, -1)$.
17. $(2, 3)$, $(2\frac{1}{2}, 2\frac{1}{2})$.
18. $(\frac{1}{2}, -2\frac{1}{2})$, $(1, -2)$.
19. $(2, -2)$, $(8, 10)$.
20. $(1, 3)$, $(-\frac{4}{7}, -3\frac{2}{7})$.
21. $(3, \frac{5}{3})$, $(5, 1)$.
22. $(1, 1)$, $(2, 3)$.
23. $(0, -6)$, $(5, 0)$.
24. $(1, 1)$, $(9, -4\frac{1}{3})$
25. $(4, 3)$, $(-3\frac{1}{5}, -1\frac{4}{5})$.
26. $(1, -2)$, $(-1, 3)$.
27. $(2, 4)$, $(-2\frac{1}{2}, -2\frac{3}{4})$.
28. $(11, 7)$, $(7, 11)$.
29. $(23, 13)$, $(-13, -23)$.
30. $(13, -11)$, $(11, -13)$.
31. $(5, 3)$, $(3, 5)$.
32. $(4, 1)$, $(-1, -4)$.
33. $(6, 2)$, $(-4, -3)$.
34. $(-3, 2)$, $(6, -1)$.
35. $(-2, -6)$, $(4, 9)$.
36. $(-\frac{3}{2}, \frac{7}{5})$, $(-2, \frac{9}{5})$.
37. $(-3, 2)$. $(1, 5)$.
38. $(4, 3)$, $(-1\frac{1}{2}, -\frac{2}{3})$.
39. $(-5, -3)$, $(4\frac{3}{11}, 3\frac{2}{11})$.
40. $(-3, 1)$, $(3\frac{4}{5}, -5\frac{4}{5})$.

EXAMPLES XXI (b) (p. 133).

1. $(3, 1)$, $(-3, -1)$, $(1, 3)$, $(-1, -3)$.
2. $(4, 1)$, $(-4, -1)$, $(1, 4)$, $(-1, -4)$.
3. $(1, -2)$, $(-2, 1)$, $(-1, 2)$ $(2, -1)$.
4. $(0, 2)$, $(0, -2)$, $(3, 1)$, $(-3, -1)$.
5. $(3, 1)$, $(-3, -1)$, $(2, 3)$, $(-2, -3)$.
6. $(1, 1)$, $(-1, -1)$, $(4, -2)$, $(-4, 2)$.
7. $(3, 1)$, $(3, -1)$, $(-3, 1)$, $(-3, -1)$.

8. $(1, \frac{1}{2}), (-1, -\frac{1}{2})$.
9. $(2, 1), (-2, -1)$.
10. $(3, 2), (-3, -2)$.
11. $(7, 5), (-7, -5)$.
12. $(2, -\frac{1}{2}), (-2, \frac{1}{2})$.
13. $(-\frac{1}{3}, 2)(\frac{1}{3}, -2)$.
14. $(4, 1), (-4, -1)$.
15. $(2\frac{1}{2}, 1\frac{1}{4}), (-2\frac{1}{2}, -1\frac{1}{4})$.
16. $(4, 3), (3, 4), (-4, -3), (-3, -4)$.
17. $(2, 3), (-2, -3), (7, -12), (-7, 12)$.
18. $(1, 3), (-1, -3), (7, 5), (-7, -5)$.
19. $(4, -3), (-4, 3)$.
20. $(6, 1), (-6, -1), (\frac{2}{3}, \frac{1}{2}), (-\frac{2}{3}, -\frac{1}{2})$.
21. $(2, -1), (-2, 1), (2\frac{1}{2}, \frac{1}{3}), (-2\frac{1}{2}, -\frac{1}{3})$.
22. $(9, 5), (-9, -5), (5, 2), (-5, -2)$.
23. $(4, 2), (-4, -2), (2, -2), (-2, 2)$.
24. $(0, 1), (0, -1), (1, -2), (-1, 2)$.
25. $(2, 1), (-2, -1), (\frac{11}{6}, \frac{19}{8}), (-\frac{11}{6}, -\frac{19}{8})$.
26. $(3, 5), (-3, -5), (1\frac{3}{2}, -\frac{1}{4}), (-1\frac{3}{2}, \frac{1}{4})$.
27. $(2, -1), (-2, 1), \left(\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.
28. $(6, 2), (-6, -2), (2\sqrt{2}, -4\sqrt{2}), (-2\sqrt{2}, 4\sqrt{2})$.
29. $(2, 1), (-2, -1), \left(\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.
30. $(4, 5), (-4, -5), (3\sqrt{3}, \sqrt{3}), (-3\sqrt{3}, -\sqrt{3})$.

EXAMPLES XXI (c) (p. 135).

1. $(\frac{3}{4}, \frac{2}{3})$.
2. $(4, 7), (-2, -2)$.
3. $(2, 4)(12\frac{1}{2}, -10)$.
4. $(1, 4), (-2, 1)$.
5. $(5, 8), (-8, -5)$.
6. $(4, 3), (-9, -\frac{4}{3})$.
7. $(\frac{2}{3}, -10\frac{1}{2}), (7, -1)$.
8. $(3, 1), (-5, -3)$.
9. $(3, -2), (4\frac{1}{4}, -7)$.
10. $(7, 4), (4, 7)$.
11. $(4, -1), (2, -2)$.
12. $(2, 3), (3, 2), (-2, -3), (-3, -2)$.
13. $(4, -1), (-3, \frac{4}{3})$.
14. $(3, -3), (-3, 3)$.
15. $(3, 2), (\frac{11}{6}, \frac{9}{8})$.
16. $(2, 2), (-2, -2), (4, 1), (-4, -1)$.
17. $(4, -3), (-4, 3), (3, -4), (-3, 4)$.
18. (6, 5).
19. (3, 5), (5, 3).
20. (3, 1), (1, 3), (2, 2).
21. 6, 5.
22. 5, 2.
23. $7\frac{1}{2}$ in., 4 in.
24. 6, 2.
25. 9, 6.
26. $(\frac{1}{3}, \frac{6}{5})$ (or $\frac{5}{6}$, 3).
27. $\frac{3}{4}$, 2.
28. $\frac{1}{2}, \frac{2}{3}$.
29. $2\frac{1}{2}$ in., $1\frac{1}{2}$ in.
30. (8 ft. by 9 in.), or (12 ft. by 6 in.).
31. $2\frac{1}{2}$, 2.
32. 6, 2.
33. 16 ft.

TEST PAPERS.

1.

1. 20, 11, 9. 2. 2, 6. 3. $13a + 12b$.
4. $67x$ pence. 5. $80x$/-, $10y$/-.

2.

1. 22, 20, 172. 2. $30g$, $50g$. 3. $27p - 17q$.
4. $(112 + x)$ lb. or $\left(8 + \frac{x}{14}\right)$ stones.
5. $(8m + 5w + 2b)$ shillings.

3.

1. (i) 22, (ii) 0, (iii) 2. 2. £$160x$. 3. $14a - 19b + 10c$.
4. $3x + 4y + 7z$. 5. $(x + y)$ degrees.

4.

1. 13, 7, 4. 2. $4x^3 - 7x^2 + 2x - 18$. 3. $7a^3 - 3a^2 - 10a + 4$.
4. $\frac{a}{d}$. 5. £$(275 + 6x)$, £$(275 - 2x)$.

5.

2. (i) $-4t$, (ii) $3b$. 3. 2. 4. $12b - 3a$. 5. $(x - \frac{11}{4})$ gallons.

6.

1. 3, 69, 8, 19. 2. $-2abc$. 3. $-15p^2 - 2q^2 - 2r^2$.
4. (i) $-4a + 2b + 4c$; (ii) $-22y + 30z$. 5. $17x$ oz., $\frac{17}{16}x$ lb.

7.

1. 0. 2. $-10w^2 + 7w - 9$. 3. $2x^3 - 18x^2 + 10x + 6$.
4. $3y^2 + 13y + 84$. Rem. 502. 5. $(54m + 30w)$ shillings.

8.

1. (i) z; (ii) x^3; (iii) $-3x^3$: (iv) $-37a - 16b$.
2. $-2x^3 + 2x^2 + 3x - 13$. 3. $2x^4 - 15x^3 + 10x^2 + 16x - 6$.
4. $3m^2 + 2m + 9$. Rem. 10. 5. $\frac{3n}{4}$ ft.

9.

2. $\frac{m}{6}$; x^7; $9s^2 - 14t^2$.
3. Perimeter $(4x + 14)$ ch., area $(x^2 + 7x + 10)$ sq. ch.
4. $x^2 - 3x - 2$. 5. $(40 - 3x)$/-.

10.

1. 659. 2. (i) $36x^{14}$; (ii) $6b^{11}c^4$.
3. $-p+3q+10r$; $3p-4q-15r$.
4. $x^5+8x^4+x^3+6x^2-20x-32$.
5. $(4x^2-x)$ sq. in.; perimeter $(21x-1)$ in.

11.

2. (i) $-7x^2+20x-32$; (ii) $-8x-12y$.
3. $x^5-3x^3+15x^2-26x+12$.
4. x^2-2x-3. Rem. -24. 5. $4\frac{1}{5}$; 5.

12.

1. 7, 10, $11\frac{1}{2}$, 13, $14\frac{1}{2}$, 16, 17, 19, $19\frac{1}{2}$, 22.
2. $66c$ ft., $36y$ yd., $224s$ oz., $(240p+20s+d)$ pence.
3. $3a-7b$; $4x^4-16x^3+15x^2-34x+21$. 4. 20, 4, 3. 5. 4.

13.

1. 2, -3. 2. $-x^2+14x-5$.
4. 1·9, -6, $-4\frac{1}{2}$. 5. $\hat{A}=33°$, $\hat{B}=42°$, $\hat{C}=105°$.

14.

1.

x	-5	-4	-1	0	3	5	7
y	$-8\frac{1}{2}$	-8	$-6\frac{1}{2}$	-6	$-4\frac{1}{2}$	$-3\frac{1}{2}$	$-2\frac{1}{2}$

2. $3x-13y$. 3. $14x^4-24x^3-14x^2+28x-32$.
4. (i) $\frac{1}{4}$; (ii) 24; (iii) -6. 5. $(4x+10)$ in., 15 in., 20 in.

15.

1.

x	-2	$-1\frac{1}{2}$	$-\frac{1}{8}$	0	$3\frac{3}{4}$	$4\frac{1}{8}$
y	$-1{\cdot}6$	$-0{\cdot}1$	2·4	4·4	15·4	16·9

2. (i) $16x+6y+15z$; (ii) $-7a^3+a^2+3a-10$.
3. $x^3+4x^2+17x+68$. Rem. 273.
4. $-\frac{3}{2}$, $-\frac{3}{2}$, 4. 5. $10\frac{1}{2}$ in., $4\frac{1}{2}$ in.

16.

1. (i) $8c$ stones; (ii) $\frac{h}{24}$ hr.; (iii) $7m+\frac{14}{3}w+\frac{7}{4}c$.
2. $-x^2-2x-8$. 3. $x^4-x^3+x^2-10x-3$. 4. $\frac{3}{2}$. 5. 7.

17.

1. 4. 2. (i) $\frac{3x}{2}$ stones; (ii) $(4x+2y+3z)$ shillings.
3. $x^2-2x+15$. 4. (i) -2; (ii) 6. 5. 3/-.

18.

1. (i) 4; (ii) 5. 2. (i) $C=\frac{nx}{12}$; (ii) $169+4x$.
4. (i) 7; (ii) 2. 5. 7 in.

19.

1. $\frac{13c}{5}$ tons, £$\frac{39c}{5}$. 3. $x^4+2x^3-13x^2-14x+24$.
4. (i) 76; (ii) -7. 5. $A=80°, 40°, 60°$.

20.

1. 28, 88, -47, -137. 3. c^3+2c^2-c-2. Rem. $=5$.
4. (i) 81; (ii) $\frac{16}{15}$. 5. 4.

21.

1. 4. 2. $6x$/-. 3. -2, 3.
4. 1, -2. 5. 25 cm., 20 cm.

22.

1. $90-\frac{x}{2}$. 3. (i) -3; (ii) 14.
2. a^4+a^3-a-1. 4. 4, -1.
5. 20 yd.

23.

1. $(180-2x)°$, 36°, 36°, 108°.
2. $28x-34y$; $2x^5-5x^4-7x^3+17x^2+5x-12$.
3. (i) -2; (ii) -18. 4. -3. 5. 3, 7.

24.

1. $\frac{9}{10}$. 2. $\frac{l}{108}$. 3. -5. 5. $1\frac{1}{2}$d., 2d.

25.

1. $1\frac{1}{3}$. 2. $2x^2 - 13x + 61$; R $= -289x + 184$.
3. (i) $\frac{1}{7}$; (ii) 36; (iii) $-4\frac{2}{3}$. 4. 8. 5. 60 gallons.

26.

1. 16. 2. $\frac{144l}{a}$ 3. $2\frac{1}{2}$. 4. $\frac{5}{8}$, $\frac{13}{8}$; -3. 5. 6, 10.

27.

1. $\left(560 - \frac{20x}{3}\right)$ lb. 2. $9a^2 + 4a - 20$. 5. 8, 12.
3. (i) 17; (ii) $-\frac{1}{4}$; (iii) 8.

28.

1. (i) 32, 7, 0, 5, 16, 27, 0; (ii) -128, -49, 0, 25, 32, 27, 0.
2. xd. 3. (i) $\frac{12a}{5}$; (ii) 34. 4. -9.
5. 2d., 3d.

29.

1. $\frac{100}{x}$ gal., £$\frac{25}{2x}$. 2. $x^2 + 7x + 12$. 4. 20.
3. (i) $-m^2 + 1$; (ii) 2. 5. £1 10/-.

30.

1. 4. 3. (ii) 27. 4. 2, -1. 5. 36.

31.

1. (i) $3600f$ ft.; (ii) $\frac{45}{22}$ ft.; (iii) $\frac{1}{f}$ sec.; (iv) $\frac{3}{f}$ sec.; (v) $\frac{3y}{f}$ sec.
2. $-3\frac{7}{12}$. 3. 3, 8, 3.
4. (i) $y^2 + 5y - 24$; (ii) $2x^2 + 7x - 30$. 5. Gold, £2; Diamonds, £5.

32.

1. 6, $6\frac{1}{4}$, $-7\frac{1}{2}$, $-12\frac{3}{4}$. 2. $11a - 13b + 34c$. 3. (i) 2; (ii) $9\frac{1}{2}$.
4. $8a^2 - 2ab - 15b^2$. 5. 220 2-lb. packets, 80 5-lb. packets.

33.

1. (i) $10a+b$; (ii) $100a+10b+c$. 2. 2. 3. 6, 2.
4. (i) $3a^2+20a-2$; (ii) b^2-4. 5. 18.

34.

1. -89. 2. ·44, $2\frac{1}{4}$. 4. $6t^2+12t-47$. 5. 4/-, 6/-.

35.

1. -136. 2. (i) $\frac{ln}{m}$; (ii) $-1{\cdot}78$. 3. 23.
4. $x+5y$ and $7x-3y$. 5. 17 in. by 5 in.

36.

1. (ii) 6. 2. (i) $-4{\cdot}33$; (ii) 8.
4. (i) $\frac{16}{7}$; (ii) $(a+9)(a-2)$. 5. 6.

37.

1. (i) $\frac{x}{12}$; (ii) $\frac{x}{60}$/-. 3. -4, -2.
2. $3x^2-2x-7$. 4. (i) 2; $(m+15)(m-3)$.
5. 3, 7.

38.

1. x^2+4x+2; (ii) x^2+3x-1. 2. 44, -44. 3. 5, -3, 4.
4. (i) 52; (ii) $(t-16)(t-3)$; $(w-7)(w+7)$. 5. 7 in. by 5 in.

39.

1. $\frac{269x}{60}$ pence. 2. (i) $5x^2+24x+12$; (ii) $3x^2-14x-73$.
3. 6, -2. 4. (i) $\frac{8}{5}$; (ii) $y-3$. 5. 60.

40.

1. $m^2-13m-126$. 2. $-19c$. 3. 1, -2.
4. (i) $(y-11)(y+6)$, $(z+6)(z+7)$; (ii) ± 4, 4, 7. 5. 16 in. by 9 in.

41.

1. 360. 2. (i) $45-12a$; (ii) $(3y-5)(2y+1)$
3. (i) 0, 5; (ii) 2, -17.
4. (i) $ax(ax-5)$; (ii) $(3a+7b)(3a-7b)$; (iii) $3(p+5q)(p-5q)$.
5. 16 in.

42.

1. 4, -3. 2. (ii) $\dfrac{x+3}{x+5}$.

4. (i) $3bc(1-2bc)$; (ii) $(4m+7pq)(4m-7pq)$; (iii) $(2l-7)(2l-3)$.

5. 40 years, 16 years.

43.

1. $(2l+2b-4)$ sq. ft. 2. -7, -16. 3. (i) $2(y-3)$; (ii) $\dfrac{a-6}{a-9}$.

4. $1\frac{1}{4}$, $\frac{1}{4}$, $\frac{3}{8}$. 5. £51 6s. 8d.

44.

1. 17. 2. $(2x-3)(x+2)$

3. (i) $(2z-9)(3z+13)$; (ii) $\begin{cases} w = \frac{7}{4} \text{ or } -1 \\ x = \frac{7}{2} \text{ or } -2 \end{cases}$.

4. (i) $\begin{cases} (x+y)(a-2b) \\ (10b+3c)(4b-7c) \end{cases}$; (ii) $\dfrac{2y-3z}{x}$.

5. 24 shillings, 28 threepences.

45.

1. £1 6/8. 2. 2·10. 3. 4, 9 and -6, 29. 5. 30 in.

46.

1. (i) 6; (ii) $3\frac{2}{3}$. 2. 1·03 or $-1·31$. 3. $(3s+2)(s-2)$.

4. (i) $\frac{1}{5}$; (ii) $q^2(2p^2+q^2)$. 5. 30.

47.

1. $2x+y=l$. 2. -1, 2. 3. (i) 1, -16; (ii) $x^2+x-6=0$.

4. (i) $(7+t)(3-t)$; (ii) $(y+24)(y-3)$; (iii) $4(2a-5b)(7a+2b)$.

5. A, £1 10/-, B, £1.

48.

2. (i) 17; (ii) 4. 3. $5x^2+12x+9$; -1, $-\frac{7}{5}$.

4. (i) $2a(a+3)(a-3)$; (ii) $3(a-b)(3a+3b-4)$.

5. 22 pence, 11 shillings, 17 sixpences.

49.

1. $x=2\frac{1}{2}$, $1\frac{3}{4}$, 1, $3\frac{1}{4}$, 4, $2\frac{3}{4}$; $y-3\frac{1}{3}$, 2, $\frac{2}{3}$, $4\frac{2}{3}$, 6, 3.

2. (i) $x^5-2x^4+5x^3-6x^2+5x-6$; Quot. $=x^3-5x^2+22x-82$: Rem. $=295x-170$. 3. 15.

4. (i) $(9+4m^2)(3+2m)(3-2m)$; $3y(2x+y)$; (ii) $\dfrac{b^2}{a^2x^2}$.

5. 5 in., 5 in., 2 in.

50.

1. $\frac{1760}{n-1}$ yd. 2. 4. 3. 3, -8.
4. $(3x+\frac{1}{2}y)(3x-\frac{1}{2}y)$; $(8x-9)(4x-9)$. 5. 18 half-crowns, 5 florins.

51.

1. (i) $\frac{2640}{\pi n}$. 2. $\frac{2}{3}$, $-2\frac{1}{2}$.
3. (i) $x(x+1)(x-2)$; (ii) $(v+2)(2v-5)$.
4. (i) $2a+1$; (ii) $\frac{1}{x-1}$. 5. 4.

52.

1. $£\frac{1}{3}(28d-3C)$; £28. 2. (i) 3, $-8\frac{1}{2}$; (ii) (1) 4, $-4\frac{1}{3}$, (2) 0, 3.
3. (i) $(a+b)(a-b)(2x-y)$; (ii) $(2x-3)(x+1)(2x^2-x-5)$.
4. $\frac{2x}{x+4}$. 5. 24 ft. by 6 ft.

53.

1. 5, -5. 2. (i) $\frac{2}{3}$, -4; (ii) 2.
3. (i) $(2a-3)(x-y+2)$; (ii) 2 cm. 4. $a-4$. 5. 2, 3.

54.

1. 2, 3. 2. (i) $t(t+7)$. 3. $\frac{2a-1}{7a(2a+3)(4a^2+9)}$.
4. $a^2(x+b)(x-b)$. 5. 3 inches.

55.

2. (i) $(x+2)(x-7)$; (ii) $(28a+9b)(8a-21b)$. 3. $\frac{3x+7}{4x+5}$.
4. (i) $xy(x+y)(x-y)$; (ii) $\frac{7w-3}{w(w-1)}$. 5. 6 ft., 4 ft.

56.

1. (i) $1\frac{1}{2}$, -1. (ii) $\frac{1}{6}$, -6. 2. $\frac{2a}{a+7}$. 3. (i) $(x+7a)(x-4a)$ (ii) $m(m+7)(m-1)(5m+6)$.
4. $(c^2-4)^2$. 5. 4 ft.

57.

1. $-1{\cdot}5,\ -{\cdot}6,\ 0,\ 1{\cdot}2,\ 1{\cdot}8;\ -15;\ 0,\ -3.$
2. (i) $(x+5)(x+6)(x-8)(x-9)$; (ii) $(23a-7b-11c)(17a-25b-5c)$.
3. $\frac{a}{5}$.
4. $36y^2(5y+2)(5y-2)^2$.
5. 3 ft.

58.

1. $\frac{d}{40-2d}$, 19·6.
2. Octagon.
3. $\frac{1}{a}$.
4. $8(c-1)(c+2)(4c-1)$.
5. 7, 8.

59.

1. (i) $p=2\left(l+\frac{A}{l}\right)$; (ii) 10 in. by $4\frac{1}{2}$ in. or 9 in. by 5 in.
2. $\frac{3w}{2}-\frac{5w}{3}$; $6x^2-5x+1$.
3. $20(2x-3)(5x-3)$.
4. 1.
5. 2 miles.

60.

1. $\frac{a+b}{ab}$, $\frac{ab}{a+b}$ days.
2. 8, 2.
3. $\frac{2a+1}{a^2-4}$.
4. 5·87 or −1·87.
5. $\frac{4}{5}$ mile.

61.

1. (i) $v=\frac{6v}{v-6}$; 30; (ii) $a+b$.
2. $(x^4-81a^4)(x^2-9a^2)$.
3. $\frac{5}{1-c}$.
4. 3·37, 1·63.
5. 36 m.p.h., 48 m.p.h.

62.

1. $\frac{ac-b}{c-1}$, $\frac{a-b}{c-1}$.
2. $\frac{x^2+4a^2}{x^2-4a^2}$.
3. (i) $\frac{h(a+3b)}{4(a+b)}$; (ii) $\frac{2a+1}{(2a-3)(3a+2)}$.
4. (i) 3·30, −·30; (ii) 3, −10.
5. 10 in.

63.

1. $\frac{by-ab}{y+a}$.
2. $4(x+4y)$.
3. $\frac{2}{y-x}$.
4. 1·35 or −1·85.
5. 8 ft.

64.

2. $a-b$.
3. $c(c-2)(2c-3)(2c+3)^2$.
4. $\dfrac{3}{3a+1}$
5. ·89, −3·39.

65.

1. $h=\dfrac{720}{a(l+b)}$.
2. $\dfrac{b}{m-a}$, $\dfrac{bm}{m-a}$.
3. (i) $2(x-1)(x+3)$; (ii) $(x^2+xy+y^2)(x^2-xy+y^2)$.
4. $\dfrac{2a+1}{a+y}$.
5. (i) ·58 or −2·78; (ii) $a=2$ or 4; root $=8$ or 16.

66.

1. $(a+8)$ pence.
2. (i) $(a+b)$; (ii) $\dfrac{b}{a}$, $-\dfrac{7b}{a}$.
3. (i) $(a+b)x$; (ii) $\dfrac{x}{y}$.
4. $\dfrac{6p-8}{p^2-4p+3}$.
5. 4 in., 12 in.

67.

1. $\dfrac{bmt}{aw}$ miles.
2. (i) $\dfrac{1+3b}{a+b}$, $\dfrac{1-3a}{a+b}$; (ii) 2, $2\frac{1}{2}$; 3, $1\frac{2}{5}$.
3. $t(t+1)$.
4. $\dfrac{1}{c+3}$.
5. 20 inches.

68.

2. (i) $(x+1)(x^2+1)$; $(2x+3y)(4x^2-6xy+9y^2)$; (ii) $9(x+y)^2(x-y)^2$.
3. $\dfrac{2x+3}{x^2-1}$.
4. $\frac{1}{2}$ or -3.
5. 5 in., 5 in., 6 in.

69.

2. (i) $-\dfrac{a}{2}$; (ii) $-1\frac{1}{3}$, 3.
3. (i) $\left(\dfrac{1}{3x}+\dfrac{1}{2y}\right)\left(\dfrac{1}{3x}-\dfrac{1}{2y}\right)$; (ii) $\left(\dfrac{4}{a}+1\right)\left(\dfrac{5}{a}-1\right)$;
(iii) $\left(a+\dfrac{3}{a}\right)\left(a^2-3+\dfrac{9}{a^2}\right)$.
4. (i) $\dfrac{2pq}{p^2-q^2}$; (ii) $\dfrac{-2pq}{p^2-q^2}$
5. 15 ft. by 12 ft. by 10 ft.

70.

2. (i) $(x+2)(x-2)^2(x^2+2x+4)$; (ii) $(y-z)(x+y+z)$.

3. $\dfrac{10ab}{4a^2-b^2}$ 4. (i) $\dfrac{ac}{a^2-b^2}$, $\dfrac{bc}{a^2-b^2}$; (ii) 8, 6, or -6, -8.

5. £48, £40.

71.

2. $\dfrac{a}{2}(a+1)$.

3. (i) $y(y+7)(y-3)$; (ii) $(2x+3y+2c)(2x+3y-2c)$;
(iii) $(xy+4)(x^2y^2-4xy+16)$; (iv) $(z-b)(z^2+bz-c)$.

4. 1. 5. 2 or 4.

72.

2. $\dfrac{a^2+b^2}{b}$. 3. (i) 1; (ii) $\dfrac{4x+21}{(x+1)(x+2)(x+3)}$.

4. 2, 4. 5. 75.

73.

1. $\dfrac{c+b}{c-b}$. 2. $\dfrac{2rs}{r^2+s^2}$. 3. $\dfrac{m^2-10m+25}{3m^2-32m+64}$.

4. $(a^2-4)(4a^2-1)(a^2-2a+4)$. 5. $-\frac{4}{5}$, 3.

74.

1. $\mathrm{P}=\dfrac{100\mathrm{A}}{rn+100}$; $n=\dfrac{100(\mathrm{A}-\mathrm{P})}{\mathrm{P}r}$.

2. (i) $2q(r-s)$; (ii) $(5a+b)(3x-2y)$; (iii) $(3a^2-2a+5)(a^2+4a-3)$;
(iv) $(x+2)(x-2)(x+5)(x-5)$.

3. $\dfrac{2b}{1-b^2}$. 4. 5·65, ·35. 5. 16 min.

75.

2. (i) $\dfrac{2a}{a+2}$; (ii) $\dfrac{p^2+pq+q^2}{p+q}$, $-\dfrac{pq}{p+q}$ 3. $-\dfrac{pq}{rs}$.

4. 2. 5. 2 in.

76.

2. $\dfrac{2b}{f+g}$, $\dfrac{f-g}{2}$. 3. (i) $(a+b-1)(a-b+1)$;
(ii) $(z+1)^2(z-1)$

4. $\dfrac{a^2}{a^2-b^2}$. 5. $-\frac{4}{11}$.

77.

1. $\frac{c}{b+1}$dy., $\frac{720}{b+1}$dy. 2. (i) $1\frac{1}{2}$; (ii) $11\frac{2}{3}$, $-\frac{3}{4}$.
3. (i) $(a-c)(t+1)(t-1)$; (ii) $(2t+a)(2t+b)$.
4. (i) $x^2+9x+20$; (ii) $\frac{x-1}{2}$. 5. 5 or $2\frac{4}{7}$.

78.

2. (i) 1, 1; (ii) 2·1, −1·8. 3. (i) $(x-y+4)(x-y-4)$; (ii) c^2+d^2.
4. 5, −3. 5. 20, 30.

79.

2. $\frac{a^2-b^2}{3ab}$, $\frac{2a^2+b^2}{3b}$. 3. 4.
4. (i) $a^2(a-x)$; (ii) $2(c+1)(b+c+1)$. 5. −1.

80.

3. 14, 49, 36. 4. $\frac{4x+40}{x^4-16}$. 5. 3 ft., 1 ft.

81.

1. $\frac{b-a^2}{ab-1}$, $\frac{a-b^2}{ab-1}$.
2. (i) $(y-1)(y-2)(y+2)(y-4)$; (ii) $2(a+1)(a+2)(a-3)$.
3. 5, −11. 4. $\frac{3}{(x-1)^2(x^2+2x+3)}$ 5. 18 hr.

82.

1. 40. 2. $3\frac{3}{4}$; $\frac{uv-1}{u+v-2}$. 3. $-\frac{2x+8}{x+3}$.
4. $2(x+1)(x-1)(2x+1)$; 1, −1, $-\frac{1}{2}$. 5. 7.

83.

1. $x^5+x^4+3x^2-x+2$. 2. 3, −1.
3. (i) $(x+2y-2)^2$; (ii) $\frac{2(a-x)(a^2+x^2)}{ax^2}$.
4. 4. 5. 40, 10, 8.

84.

1. $\frac{1}{n}$. 2. 1, $\frac{3}{a}$. 3. $\frac{q}{p}$. 4. 0, 2. 5. $3\frac{1}{2}$ m.p.h.

85.

1. (i) 15·514; (ii) ·37 (approx.); (iii) 1, −1. 2. $m-1$, $l-1$.
3. (i) $3a(a-2b)(a^2+2ab+4b^2)$; (ii) $(x+1)\{(b-c)x+a\}$.
4. (i) $3\frac{1}{3}$, $-\frac{3}{5}$; (ii) 4·89, ·61. 5. 4 m.p.h., 20 min.

86.

1. $10mx$; $\frac{10mx}{by}$. 2. $\frac{1}{2}$, 3; $-\frac{13}{10}$, 11. 3. 1. 4. $1\frac{1}{2}$.

5.

A	92	374
B	187	276

87.

1. x^3-2x. 2. $c-b$. 3. $\frac{2(y+z)}{x+y+z+w}$.
4. (i) $1\frac{3}{8}$, $-1\frac{3}{8}$; (ii) 2, −. 5. 12 m.p.h.

88.

1. $L=\frac{l}{1-aT+at}$. 2. $20x^5(x-3)(x+8)(x^2+3x+9)$.
3. (i) $\frac{a}{a^2+b^2}$, $\frac{b}{a^2+b^2}$; (ii) $\frac{3}{a}$, $-\frac{8}{3a}$.
4. a. 5. £208, £312.

89.

1. $-\frac{3y+6}{2}$. 2. (i) $(3x^2-y)(x-3y^2)$; (ii) $(a-b)(2a-b+c)$.
3. $\frac{y}{2x^2}$. 4. $-\frac{ab}{a^2-b^2}$. 5. 10 ft. by 8 ft.

90.

2. $-\frac{x+2y}{x-2y}$.
3. (i) $(2a+c)(2a-c)(x-y)$; (ii) $(c-2a)(x-y)(x^2+xy+y^2)$.
4. $\frac{7}{10}$, −3. 5. $2a-2$, $2a$, $2a+2$.

91.

2. 1, 0, 2. 3. (i) $2abc(a+b+c)$, (ii) $(a-b)(a+b)^3$.
4. $7\frac{1}{2}$. 5. 3/3; 4/5.

92.

2. $m = 3$, $x = 5$; $x = -2\frac{1}{2}$. 4. 100, 250. 5. 1·82, −·82; $-\frac{1}{2}$.

93.

2. $-2\frac{1}{2}$, $-3\frac{1}{2}$, $5\frac{1}{2}$. 3. $\frac{1}{ab}$. 4. 33 yr., 12 yr. 5. 2·35.

94.

2. $(a+b)^2$, $(a-b)^2$.
3. (i) $a^2b^2(a^2+b^2)(a^2-2ab-b^2)$. (ii) $(x-1)(2x+1)(3x+2)$.
4. A(12/6), B(7/6). 5. −2·25; 1·3 and −2·3.

95.

2. 2, $\frac{8}{5}$; 2, 3, −5. 3. 1. 4. 143. 5. 9·5; 1·57, 3·52.

96.

1. $P = m(u^2+v^2)$; $Q = 2muv$; $\frac{P}{Q} = \frac{x^2+1}{2x}$.

2. $y = 5$ or $\frac{1}{5}$; $x = -5\frac{2}{3}$ or $-1\frac{2}{5}$.
3. (i) $(x-1)(3x+1)(5x^2+2x+1)$; (ii) $(1+x)(a^2+b^2x)$.
4. 85. 5. AF = FC (when $x = 3\frac{1}{2}$ in.).

97.

2. $\frac{4xy}{x+y}$. 3. 4, −2, −2. 4. 7, $-\frac{30}{37}$. 5. 23·1 in.

98.

2. 2, 5.
3. $\frac{x^2+8xy-6y^2}{(x-2y)(x+3y)(x+4y)}$; $\frac{2}{c}$.
4. 44.
5. 4·24, −4·24.

99.

2. 6, 2, 1. 3. ·083, −12. 4. $k=16$; 16 ft. 5. 0·45.

100.

1. $a=\frac{pv-PV}{PVt-pvT}$. 3. 4, $1\frac{2}{3}$. 4. 2/4. 5. 0·58.

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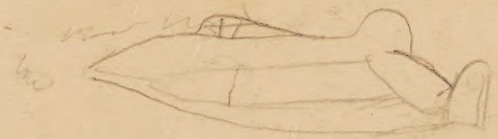

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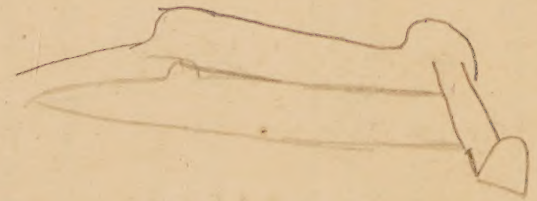

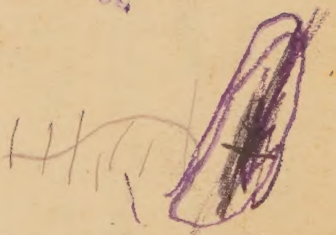

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